Effective boundary conditions for thin periodic coatings

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Introduction

L'objectif de cette thèse est de contribuer à la modélisation mathématique et numérique dans le domaine fréquentiel de la diffraction d'ondes, acoustiques ou électromagnétiques, par des obstacles recouverts par des revêtements minces qui sont fortement hétérogènes parce que leurs caractéristiques physiques varient rapidement, typiquement de façon périodique, le long du revêtement, c'est à dire parallèlement à la surface de l'obstacle.

Ce type de revêtement se rencontre dans nombre d'applications, notamment en furtivité radar où la combinaison de plusieurs types de matériaux permet d'améliorer les propriétés d'invisibilité par rapport à des ondes radars. On le rencontre aussi dans les revêtements destinés à protéger des composantes électroniques des radiations externes. Notre travail pourrait également être d'intérêt pour le contrôle non destructif de composants optiques ou nano-optiques périodiques (voir les exemples des nano-grass ou de métamatériaux Fig. 1) où la longueur d'onde utilisée pour sonder le milieu est plus grande que la périodicité du milieu.

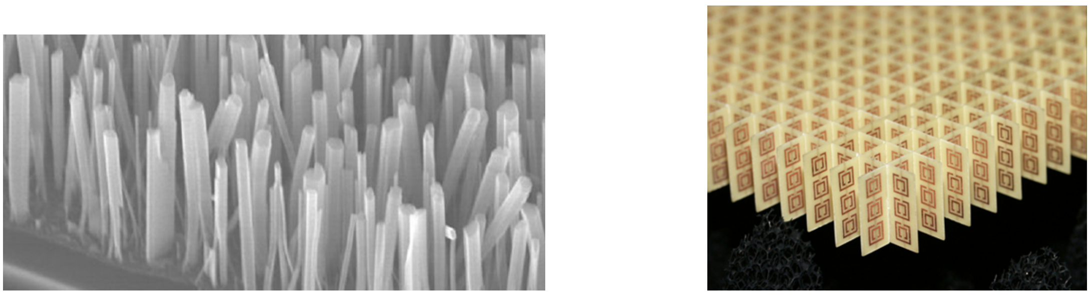

Figure 1: Exemples de "nanograss" (gauche) et de métamatériaux (droite)

Compte tenu des petites échelles spatiales mise en jeu, la modélisation numérique directe de tels phénomènes s'avère extrêmement coûteuse voire impossible (en 3D notamment). Il est alors souhaitable, dans une phase de modélisation mathématique préalable de proposer un modèle approché consistant à remplacer la présence du revêtement mince par une condition aux limites dite équivalente ou effective. De telles conditions sont communément appelées dans la littérature conditions d'impedance généralisées (Generalized Impedance Boundary Conditions (GIBC's) en anglais [64]).

Si on aborde la question avec l'oeil du mathématicien, la construction préalable d'une telle condition aux limites approchée s'appuie tout naturellement sur un développement asymptotique de la solution recherchée par rapport à la petite échelle δ qui représente à la fois l'épaisseur de la couche mince et la période du revêtement.

Dans certaines asymptotiques, résolument différentes de celles considérées ici, le petit paramètre de l'analyse est la longueur d'onde lambda, qui peut le cas échéant être proportionnel au paramètre δ [20, 24, 52, 53, 8]. Ce n'est pas cette situation que nous considérons ici : la longueur d'onde restera grande devant l'échelle δ.

De nombreux travaux ont déjà été effectués dans cette direction. Dans le cas de revêtements à caractéristiques physiques homogènes, la littérature est abondante : voir par exemple [40, 14, 13, 2, 42, 11] et [44, 39, 17, 27] et les références qu'elles contiennent. On s'appuie typiquement sur des techniques de "rescaling" par rapport à la cordonnée normale à la surface de l'obstacle, des equations à l'intérieur de la couche mince.

Le cas de revêtements à caractéristiques périodiques a également été abordé dans la littérature : voir par exemple [65, 7, 62, 61, 37, 32, 56, 30, 23] et les références qu'elles contiennent pour le problème scalaire et voir [38, 36] pour le problème de Maxwell. Dans ce cas, le problème est beaucoup plus difficile d'un point de vue technique car le comportement de la solution est par nature hautement multi-echelle et il faut donc combiner différents types de développements asymptotiques : typiquement - c'est le choix qui sera fait dans cette thèse - on s'intéressera à combiner des techniques d'homogénéisation dans la couche mince, avec des développements réguliers en puissances de δ loin de cette couche, le couplage se faisant par une méthode de développements asymptotiques raccordés [48, 49]. Les résultats disponibles dans la littérature sont d'une certaine façon incomplets car limités à des geometries simples (cylindriques [37] ou 2D [61, 7]) voire à des equations simplifiées [56, 30, 23]). Par ailleurs les travaux cités précédemment s'intéressent rarement au développement asymptotique complet de la solution, lequel permet pourtant a priori de construire une hiérarchie de conditions d'impedances de plus en plus précises.

Cette thèse a notamment pour but de combler ces lacunes en attaquant le problèmes de revêtements périodiques sur des surfaces 3D quelconques (régulières toutefois) et ce, tant pour les ondes acoustiques (équation de Helmholtz) que pour les ondes électromagnétiques (equations de Maxwell en régime harmonique). Notons que le problème présente des difficultés nouvelles et substantielles :

1. Du point de vue conceptuel, la définition de fonctions périodiques le long d'une surface quelconque est loin d'être évidente et certainement non intrinsèque. Nous apporterons une réponse possible, qui nous parait raisonnable vis a vis des applications, dans le premier chapitre de la thèse : la notion de revêtement périodique ne fait plus référence seulement à la géométrie de la surface de l'obstacle mais à un fonction supplémentaire censée expliquer la conception du revêtement périodique.

2. Du point de vue purement technique, la tâche est sensiblement plus délicate dans la mesure où, à la manipulation des techniques de développements de type multi-échelles il faut adjoindre le maniement d'outils de la géométrie différentielle.

Du point de vue de la démarche, nous nous sommes très largement inspirés des travaux de thèse de Bérangère Delourme [34] et notre travail, qui se veut complet du point de vue scientifique, respectera les étapes suivantes :

a. Etablissement d'un développement asymptotique formel de la solution du problème exact.

b. Justification mathématique de ce développement au travers d'estimations d'erreur.

c. Construction de conditions aux limites approchées.

d. Etude de la stabilité des problèmes approchés.

e. Etude mathématique de l'erreur entre la solution exacte et la solution approchée.

f. Approximation numérique des modèles approchés et validation numérique de ces modèles.

La thèse comporte trois parties. Les deux premières sont consacrées à l'équation de Helmholtz scalaire, la dernière aux equations de Maxwell.

La partie I est dédiée au développement asymptotique de la solution du problème de Helmholtz avec revêtement mince périodique.

Au chapitre 1, nous présentons le problème dit exact qui fera l'objet d'une analyse asymptotique. Nous introduisons notamment la notion de Ψ_Γ périodicité qui permet de donner un sens à la notion de fonction périodique le long d'une surface. Comme déjà dit plus haut, cette notion fait implicitement référence à un processus de fabrication des revêtements minces. Elle est de ce fait non intrinsèque.

Au chapitre 2, très technique et calculatoire, nous établissons le développement asymptotique multi-échelle formel de la solution du problème exact. Nous établissons également par récurrence l'existence des termes de ce développement.

Le chapitre 3 est consacré à la justification rigoureuse du développement asymptotique établi au chapitre 2, ce qui passe par l'étude préliminaire de la stabilité par rapport au petit paramètreδ de la solution du problème exact.

La partie II est consacrée à la construction, la justification mathématique et l'approximation numérique de conditions d'impédance approchées. En l'occurrence nous nous limitons aux conditions d'ordre 1 et 2 par rapport au petit paramètre δ.

Au chapitre 4, nous construisons et analysons ces conditions approchées. Cela passe par une représentation plus explicite (par rapport à la façon dont ils sont définis au chapitre 3) des trois premiers termes du développement asymptotique (sections 4.3 et 4.4) et l'analyse de la stabilité des problèmes approchés (section 4.5).

Dans les chapitres 5 à 7, dédies au calcul numérique, nous nous sommes limités à la dimension deux.

Le chapitre 5 est consacré aux aspects numériques de l'exploitation des conditions d'imédance ce qui passe notamment par la détermination préalable des coefficients effectifs apparaissant dans ces conditions, ce qui amène à résoudre des problèmes dits problèmes de cellule (phase de pré-traitement)

Le chapitre 6 s'écarte un peu du droit fil de la démarche : il est simplement consacré à la présentation d'une méthode d'éléments finis fiables pour le calcul d'une bonne approximation de la solution exacte. Cette étape est nécessaire pour valider numériquement les modèles approchés ce qui est l'objet du chapitre suivant. Cette présentation s'accompagne d'une étude théorique prenant en complet la co-existence de deux petits paramètres géométriques : la petite échelle δ et le pas de maillage h.

Comme annoncé plus haut, le chapitre 7 est destiné à la validation numérique des modèles approchés en utilisant les méthodes utilisées aux chapitres 5 et 6. Il s'agit notamment de vérifier que les expériences numériques sont cohérentes avec les estimations d'erreur du chapitre 4.

La partie III est consacrée aux ondes électromagnétiques. Elle s'appuie notamment sur les notions du chapitre 1 et consiste essentiellement à reprendre la démarche des deux premières parties de la thèse (à l'exception des aspects numériques qui n'ont pu être abordés faute de temps) dans le cas plus difficile des équations de Maxwell 3D.

Ainsi, après une brève présentation du problème étudié au chapitre 8, le chapitre 9 - équivalent du chapitre 3 - est consacré au développement asymptotique formel de la solution.

Le chapitre 10 est consacré à la justification rigoureuse de ce développement. II faut noter qu'un des points particulièrement délicat de l'analyse est l'étude de la stabilité de la solution du problème approché (section 11.4).

Enfin, au chapitre 11, pendant du chapitre 4, nous établissons et justifions rigoureusement une première condition d'impédance approchée, dite d'ordre 1.

Contents

Part I

The case of Helmholtz equation and modeling

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Chapter 1

Presentation of the problem

As we said in the introduction of the thesis, our purpose is to find "equivalent" or "effective" boundary conditions for the diffraction of electromagnetic waves by an object having a regular surface Γ with a thin periodic coating, i. .e a thin layer C^δ (the meaning of δ is explained hereafter), of ferromagnetic material for instance, sticked of the boundary of a perfectly reflecting object O. This is the model problem presented in section 1.1. More precisely, we are interested in the case where the thin layer is highly heterogeneous in the sense that its physical characteristics oscillate periodically along the surface.

One difficulty in the case of a general function is to give a precise sense to the above notion of periocicity. The approach that we follow in sections 1.2 is the following:

- First, we give in section 1.2.1 a rigorous mathematical definition to a periodic function along the on the surface Γ when Γ can be described as the image of a part of a plane by some smooth transformation. More precisely, we assume the existence of $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ and say that a function f is periodic on Γ if there exists a periodic function $\hat{f} : \mathbb{R}^2 \mapsto \mathbb{R}$ i. e. for all $(m, n) \in \mathbb{Z}^2$ and $(x, y) \in \mathbb{R}^2, \hat{f}(m + x, n + y) = \hat{f}(x, y))$ such that $f = \hat{f} \circ \psi_\Gamma$. Note that this definition is not intrinsic .
- We will extend in section 1.2.2 this definition for function definintion on the thin coating C^δ.
- We give illustrate in section 1.2.3 this defintion in the case of classical geometries(torus, sphere, cylinder).
- The above description is in general (in fact as soon as Γ does not have the same topology as the torus) not possible globally but only piecewise (in local zones, called periodic zones).(section 1.2.4) To overcome this difficulty we propose a slight modification of our definition that relies on a specific treatment of the transition regions between periodic zones.(section 1.2.5)

Next we reformulate in section 1.3, the diffraction problem in order to facilitate the method that we shall use for the aymptotic analysis in δ, namely a combination of matched asymptotics and homogenization. Since this will rely on the use on local coordinates at the neighborhood of Γ, we reformulate the original exterior problem in $\mathbb{R}^3 \setminus O$, into an equivalent one posed in a fixed tubular neighborhood of Γ (see [19, **2.7.** Normal

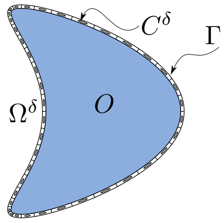


Figure 1.1: Illustration of the geometry

Bundles and Tubular Neighborhoods] for instance), denoted Ω, in which local normal and tangential coordinates can be used (as in [14, 17, 45] for instance). The inner surface of Ω is Γ and whose outer surface is treated via a transparent boundary condition. This condition is abstract and in general non explicit but, this s the important fact for the anaysis, it is independent of δ.

1.1 The model scalar problem

Let us start with a quick description of the geometry of our problem and a presentation of the model problem.

Let O be a bounded domain of $\mathbb{R}^3$ such that $\mathbb{R}^3 \backslash O$ is connected with regular boundary Γ and let $\delta > 0$. We call the "thin coating of width δ" the following subset C^δ of O:

$$C^\delta := \{x \in O,\ \operatorname{dist}(x, \Gamma) < \delta\}.$$

Here the quantity $\operatorname{dist}(x, \Gamma)$ is the distance of x from the surface Γ defined by

$$\operatorname{dist}(x, \Gamma) := \inf_{x_\Gamma \in \Gamma} |x - x_\Gamma|,$$

and $|.|$ is the classical Euclidean norm of $\mathbb{R}^3$. We need to introduce the complement of O in $\mathbb{R}^3$ $\Omega := \mathbb{R}^3 \setminus \overline{O}$ and $\Omega^\delta := \overline{\Omega} \cup C^\delta$. We refer the reader to the Figure 1.1 for an illustration in 2D. The problem that we are interested in is the following: Find $u^\delta \in H^1_{\text{loc}}(\Omega^\delta)$ such that:

$$\begin{cases} \operatorname{div}\big(\rho^\delta\ \nabla u^\delta\big) + k^2 \mu^\delta\ u^\delta = f, & \text{in } \Omega^\delta, \\ \partial_{n^\delta} u_\delta = 0 & \text{on } \partial\Omega^\delta, \end{cases} \tag{1.1.1}$$

and u_δ satisfies the Sommerfeld radiation condition:

$$\lim_{R\to\infty} \int_{|x|=R} |\partial_r u^\delta - iku^\delta|^2 = 0.$$

Here n^δ and n are the outward unit normal vectors to $\partial\Omega^\delta$ and Ω respectively, $k \in \mathbb{R}$ is the wave-number and f denotes a given source term.

Moreover ρ^δ, μ^δ denote the acoustical characteristics of the medium supposed to be equal to 1 in Ω and $\delta-$ periodic in the thin coating C^δ. This is the main feature of our problem. The definition of periodicity in the thin coating C^δ will be given later.

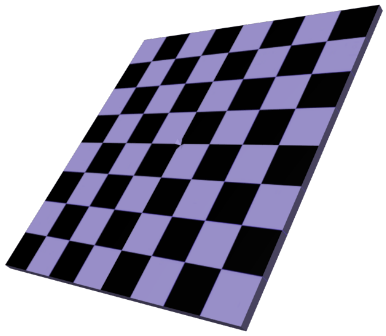


Figure 1.2: Illustration of periodic plate

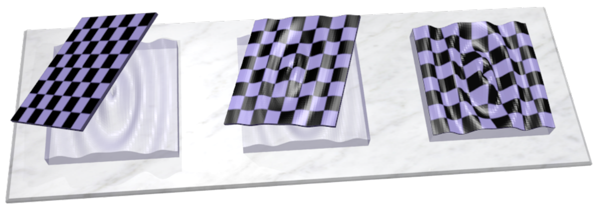


Figure 1.3: The covering process

1.2 Notion of periodic coating

The definition is inspired from practical considerations. Imagine that a manufacturer has a δ-periodic plate in the sense that the acoustic coefficients ρ, μ are δ periodic and assume that the width if this plate is δ. Figure 1.2 is an illustration of a such plate. Then the plate is deformed in order to stick on the surface of the 3D object O. This process is illustrated in Figure 1.3. This procedure is repeated until there is a covering of all the surface ∂O.

1.2.1 Definition of periodic functions on Γ

Let $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ be a given function defined from the surface of the object $\Gamma = \partial O$ into the plane $\mathbb{R}^2$. Intuitively this function represents the inverse of the deformation of the plane shown in Figure 1.2 and Figure 1.2.1. Thanks to ψ_Γ we can give a first definition of periodicity for functions defined on the surface Γ. Let us emphasize that the notion of periodicity that we introduce is relative to a family of functions depending on the small parameter δ, not to a single function.

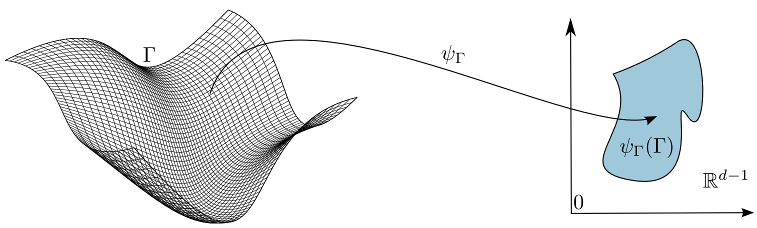


Figure 1.4: The application ψ_Γ

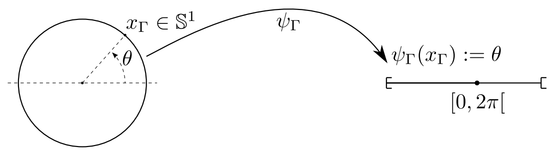


Figure 1.5: The circle

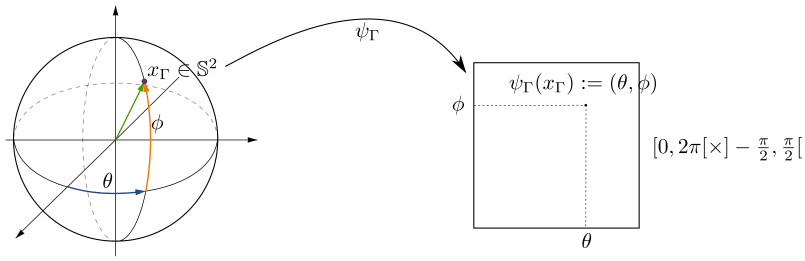


Figure 1.6: the sphere

Definition 1.2.1 (ψ_Γ $-\delta-$periodicity on a surface Γ). *Let $u^\delta : \Gamma \mapsto \mathbb{R}$ be a sequence of functions indexed by δ and defined on the surface Γ. We say that u_δ is ψ_Γ $-\delta-$periodic if there exists a reference function $\hat{u} : \mathbb{R}^2 \mapsto \mathbb{R}$, 1-periodic i.e.*

$$\hat{u}(\hat{x}_1 + \hat{m}_1, \hat{x}_2 + \hat{m}_2) = \hat{u}(\hat{x}_1, \hat{x}_2), \ \forall \big((\hat{x}_1, \hat{x}_2), (\hat{m}_1, \hat{m}_2) \big) \in \mathbb{R}^2 \times \mathbb{Z}^2,$$

such that we have for all $x_\Gamma \in \Gamma$:

$$u^\delta(x_\Gamma) = \hat{u}(\hat{x}) \quad \textit{and} \quad \hat{x} := \frac{\psi_\Gamma(x_\Gamma)}{\delta}.$$

Remark 1.2.2. *If $\Gamma = \mathbb{R}^2$ and ψ_Γ is the identity then the ψ_Γ $-\delta-$periodicity is equivalent to the classical $\delta-$periodicity i.e.*

$$u^\delta(\hat{x}_1 + \delta\hat{m}_1, \hat{x}_2 + \delta\hat{m}_2) = u^\delta(\hat{x}_1, \hat{x}_2), \ \forall \big((\hat{x}_1, \hat{x}_2), (\hat{m}_1, \hat{m}_2) \big) \in \mathbb{R}^2 \times \mathbb{Z}^2$$

Thus the ψ_Γ $-\delta-$periodicity is a generalization of classical $\delta-$periodicity for curved surface. Let us give some examples of ψ_Γ for some simple geometries.

- For the circle $\Gamma = \mathbb{S}^1 := \{x \in \mathbb{R}^2, \ |x| = 1\}$ we can choose ψ_Γ defined for $x \in \Gamma$ by $\psi_\Gamma(x) := \theta$ where θ is the unique solution in $[0, 2\pi[$ of $x = (\cos\theta, \sin\theta)$. (See figure Figure 1.5)

- For the unit sphere $\Gamma := \mathbb{S}^2 := \{x \in \mathbb{R}^3, \ |x| = 1\}$ we can choose ψ_Γ defined for $x \in \Gamma$ by
$$\psi_\Gamma(x) := (\theta, \phi), \tag{1.2.2}$$
where (θ, ϕ) is the unique solution in $[0, 2\pi[\times[-\frac{\pi}{2}, \frac{\pi}{2}[$ of $x = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$.(See Figure 1.6)

- For the torus $\Gamma := \{((R + r\cos u)\cos v, (R + r\sin u)\cos v, r\sin v)\}$ we can choose ψ_Γ defined for $(x_1, x_2, x_3) \in \Gamma$ by $\psi_\Gamma(x_\Gamma) := (u, v)$ where (u, v) is the unique solution

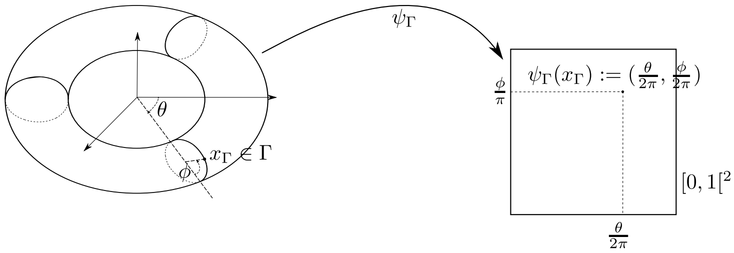


Figure 1.7: The torus

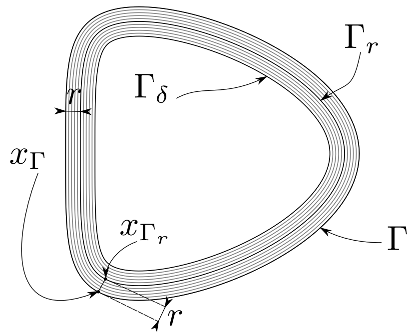


Figure 1.8: The partition $(\Gamma_r)_{0<r<\delta}$ of C^δ and the point x_Γ associated to the point x_{Γ_r}

in $[0, 2\pi[^2$ of

$$\begin{cases} x_1 = (R + r\cos u)\cos v, \\ x_2 = (R + r\sin u)\cos v, \\ x_3 = \sin v. \end{cases}$$

We refer the reader to Figure 1.7 for this example.

1.2.2 Extension of $\psi_\Gamma - \delta-$periodicity for functions defined on C^δ

One can extend the 2D $\psi_\Gamma - \delta-$periodicity to 3D by taking advantage of the function ψ_Γ. Indeed, one interpret a thin coat as a superposition of 2D surfaces. More precisely we have for all $\eta_0 \geq \delta > 0$ the following partition of C^δ: (η_0 is the real number which appear in (1.2.3))

$$C^\delta = \bigcup_{0<r<\delta} \Gamma_r,$$

where we define for $r > 0$ the surface $\Gamma_r := \{x \in O, \operatorname{dist}(x, \Gamma) = r\}$. This partition is illustrated in Figure 1.8. Now let us construct from the function $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ a new function $\psi_{\Gamma_r} : \Gamma_r \mapsto \mathbb{R}^2$. First we recall from [19, **2.7.** Normal Bundles and Tubular Neighborhoods] that the following result holds:

Proposition 1.2.3. *If Γ is a C^2 surface($d = 3$) or curve ($d = 2$) then there exists $\eta_0 > 0$ such that for all $x \in \mathbb{R}^3$ if $\operatorname{dist}(x, \Gamma) \leq \eta_0$ then the minimizer of the functional $x_\Gamma \mapsto |x - x_\Gamma|$ is unique.*

Assume that $r < \eta_0$, then thanks to Proposition 1.2.3 we can define the function $\psi_{\Gamma_r} : \Gamma_r \mapsto \mathbb{R}^2$ for $x_{\Gamma_r} \in \Gamma_r$ by:

$$\psi_{\Gamma_r}(x_{\Gamma_r}) := \psi_\Gamma(x_\Gamma),$$

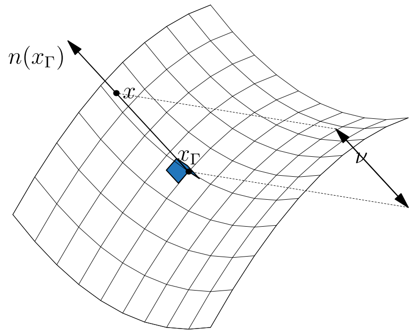


Figure 1.9: Illustration of local coordinate

where x_Γ is the unique minimizer of the functional $x_\Gamma \mapsto |x - x_\Gamma|$. We refer the reader to Figure 1.8 for a graphical illustration of this last point x_Γ. Thus, we can give the first most intuitive definition of $\psi_\Gamma -\delta-$ periodicity on C^δ.

Definition 1.2.4. *We say that a sequence of function $(u_\delta)_\delta$ defined on C^δ is $\psi_\Gamma -\delta-$periodic if for all $0 < r < \delta$ the sequence of restriction $\big((u_\delta)_{|\Gamma_r}\big)_{\delta>0}$ is $\psi_{\Gamma_r} - \delta-$periodic.*

However, although this last definition is very intuitive it is not practical for our analysis. Therefore hereafter we give an equivalent characterization. First define the local coordinates mapping(See for instance [14, 39, 52, 53]) $\mathcal{L} : C_{\eta_0} \mapsto \Gamma \times] - \eta_0, 0[$ for $x \in C_{\eta_0}$ by:

$$\mathcal{L}(x) := (x_\Gamma, \nu),$$

where x_Γ is the unique minimizer of $x_\Gamma \mapsto |x - x_\Gamma|$ and $\nu := \big(x - x_\Gamma, n(x_\Gamma)\big)$. Here $n : \Gamma \mapsto \mathbb{R}^3$ is the unit outward normal. We refer the reader to Figure 1.9 for a graphical illustration of all these quantities. Thus we can state the following reformulation of the $\psi_\Gamma -\delta$ periodicity:

Definition 1.2.5. *Let $(u_\delta)_{\delta>0}$ be a sequence of functions defined on C^δ. This sequence is called $\psi_\Gamma -\delta-$periodic if and only if there exists a reference function $\hat{u} : \hat{\Omega} := \mathbb{R}^2 \times] - 1, 0[$ satisfying:*

$$\hat{u}(\hat{x} + \hat{m}, \hat{\nu}) = \hat{u}(\hat{x}, \hat{\nu}), \ \forall (x, m) \in \mathbb{R}^2 \times \mathbb{Z}^2 \ \textit{and} \ \hat{\nu} \in] - 1, 0[,$$

such that for all $x \in C^\delta$ we have:

$$u_\delta(x) = \hat{u}(\hat{x}, \hat{\nu}) \quad \textit{with} \quad (\hat{x}, \hat{\nu}) := \frac{(\psi_\Gamma(x_\Gamma), \nu)}{\delta} \ \textit{and} \ (x_\Gamma, \nu) = \mathcal{L}(x).$$

1.2.3 Example of the $\psi_\Gamma -\delta-$ periodicity for simple geometries

Here we show some examples of the $\psi_\Gamma -\delta$ periodicty for simple geometries in order to illustrate this definition.

1.2.3.1 The cylinder

The cylinder is the following set:

$$\Gamma := \Big\{ \big(\cos(\theta), \sin(\theta), z \big), \theta \in [0, 2\pi[\text{ and } z \in \mathbb{R} \Big\}.$$

In this case, one can easily show that for a point $x \in \mathbb{R}^3$ of the form:

$$x = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ z \end{pmatrix},$$

for some $(\theta, r, z) \in [0, 2\pi[\times\mathbb{R}_+ \times \mathbb{R}$ that the local coordinate map $\mathcal{L}$ is given by:

$$\mathcal{L}(x) = (x_\Gamma, \nu) \quad \text{with} \quad x_\Gamma = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ z \end{pmatrix} \quad \text{and} \quad \nu = r - 1. \tag{1.2.3}$$

See Figure 1.11 for an illustration. Thus from this we directly have that:

$$C_\delta = \left\{ r\big(\cos(\theta), r\sin(\theta), z\big), \theta \in [0, 2\pi[, 1 - \delta < r < 1 \text{ and } z \in \mathbb{R} \right\}.$$

See Figure 1.11 for an illustration.

Assume first that ψ_Γ is the map defined for $x_\Gamma \in \Gamma$ of the form $\big(\cos(\theta), \sin(\theta), z\big)$ by:

$$\psi_\Gamma(x_\Gamma) := (\theta, z). \tag{1.2.4}$$

Then in this case, thanks to (1.2.3), the $\psi_\Gamma - \delta-$periodicity is equivalent to the existence of a reference function $(\hat{x}_1, \hat{x}_2, \hat{\nu}) \mapsto \hat{\rho}(\hat{x}_1, \hat{x}_1, \hat{\nu})$ one periodic in $(\hat{x}_1, \hat{x}_2)$ such that:

$$\rho^\delta(x) = \hat{\rho}\left(\frac{\theta}{\delta}, \frac{z}{\delta}, \frac{r-1}{\delta}\right).$$

In the two following particular cases, this last property is more simple to imagine:

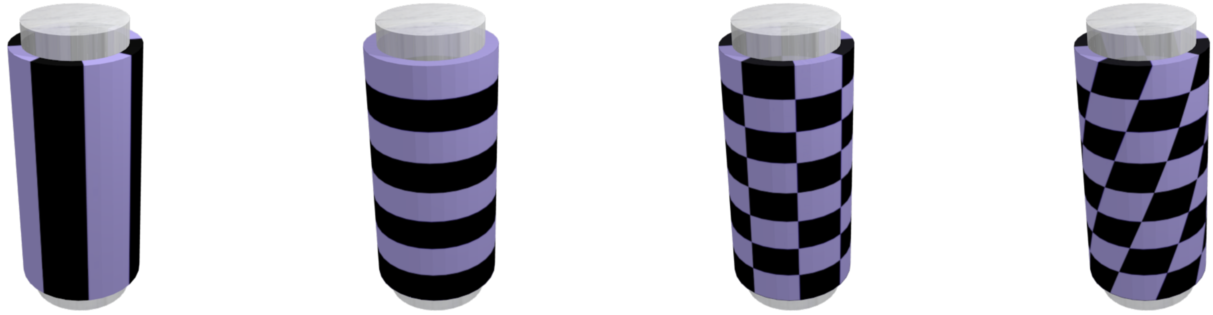
- If the reference function $(\hat{x}_1, \hat{x}_2, \hat{\nu}) \mapsto \hat{\rho}(\hat{x}_1, \hat{x}_1, \hat{\nu})$ only depends of $\hat{x}_1$ then the $\psi_\Gamma - \delta-$periodicity is equivalent to the existence of a $\delta-$periodic function $\rho_\delta : \mathbb{R} \mapsto \mathbb{R}$ such that:
$$\rho^\delta(x) = \hat{\rho}_\delta(\theta) \quad \text{and} \quad \forall \theta' \in \mathbb{R},\, \rho_\delta(\theta' + \delta) = \rho_\delta(\theta').$$
See Figure 1.10(a) for a graphical illustration of this case.

- If the reference function $(\hat{x}_1, \hat{x}_2, \hat{\nu}) \mapsto \hat{\rho}(\hat{x}_1, \hat{x}_1, \hat{\nu})$ only depends of $\hat{x}_2$ then the $\psi_\Gamma - \delta-$periodicity is equivalent to the existence of a $\delta-$periodic function $\rho_\delta : \mathbb{R} \mapsto \mathbb{R}$ such that:
$$\rho^\delta(x) = \hat{\rho}_\delta(z) \quad \text{and} \quad \forall z' \in \mathbb{R},\, \rho_\delta(z' + \delta) = \rho_\delta(z').$$
See Figure 1.10(b) for a graphical illustration of this case.

More generally, if we keep (1.2.4) as a definition of the map ψ_Γ and assume that the reference function $(\hat{x}_1, \hat{x}_2, \hat{\nu}) \mapsto \hat{\rho}(\hat{x}_1, \hat{x}_1, \hat{\nu})$ depends of the two arguments $(\hat{x}_1, \hat{x}_2)$ then the $\psi_\Gamma - \delta-$periodicity is equivalent to the existence of a $\delta-$periodic function $\rho_\delta : \mathbb{R}^2 \mapsto \mathbb{R}$ such that:

$$\rho^\delta(x) = \hat{\rho}_\delta(\theta, z) \quad \text{and} \quad \forall (\theta', z') \in \mathbb{R}^2,\, \rho_\delta(\theta', z') = \rho_\delta(\theta' + \delta, z') = \rho_\delta(\theta', z' + \delta).$$

See Figure 1.10(c) for a graphical illustration of this case.



(a) (b) (c) (d)

Figure 1.10: Example of the $\psi_\Gamma - \delta -$periodicity for the cylinder

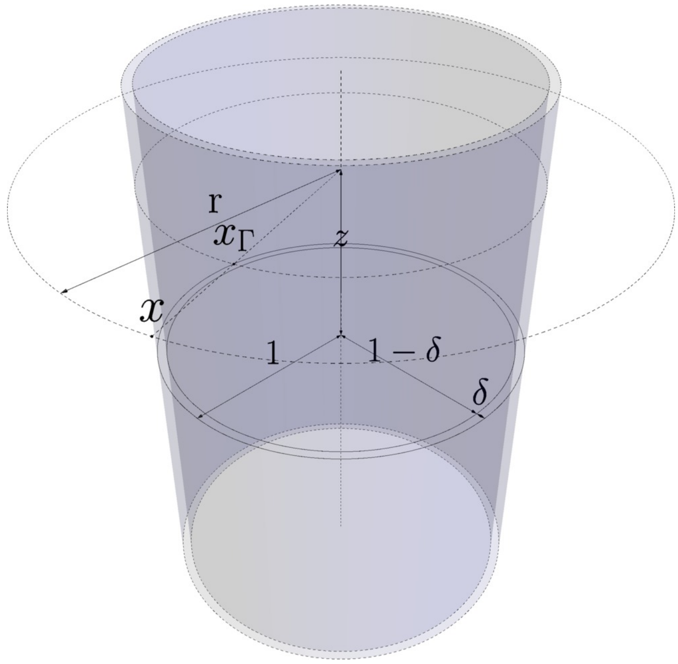


Figure 1.11: Illustration of the map $\mathcal{L}$ and the thin coat C_δ in the case of the cylinder

Now let us illustrate the dependence of the notion of $\psi_\Gamma - \delta -$periodicity with respect to the choice of the map ψ_Γ. Then we replace the definition (1.2.4) by the following one:

$$\psi_\Gamma(x_\Gamma) := (\theta, z + \eta\theta),$$

for some $\eta > 0$ and we assume that the reference function does not depend of the argument $\hat{\nu}$. Then in this case the $\psi_\Gamma - \delta -$periodicity is equivalent to the existence of a function $\rho_\delta : \mathbb{R}^2 \mapsto \mathbb{R}$ satisfaying:

$$\forall x' \in \mathbb{R}^2,\ \forall (m,n) \in \mathbb{Z}^2,\ \rho_\delta(x' + mv_1 + nv_2) = \rho_\delta(x') \quad \text{with} \quad v_1 := \begin{pmatrix} \delta \\ -\eta\delta \end{pmatrix} \quad \text{and} \quad v_2 := \begin{pmatrix} 0 \\ \delta \end{pmatrix}.$$

such that $\rho^\delta(x) = \hat{\rho}_\delta(\theta, z)$. See Figure 1.10(c) for a graphical illustration of this case.

(a) (b) (c)

Figure 1.12: Example of the $\psi_\Gamma - \delta -$periodicity for the cylinder

1.2.3.2 The sphere

The unit sphere is the following set:

$$\Gamma := \left\{ x \in \mathbb{R}^3,\ |x| = 1 \right\}.$$

One can easily show that for all $x \in \mathbb{R}^3$ we have $\mathcal{L}(x) = (x_\Gamma, \nu)$ where:

$$x_\Gamma = \frac{x}{|x|} \quad \text{and} \quad \nu = |x| - 1.$$

See Figure 1.13 for an illustration. Thus from this, we directly have the following characterisation if the thin coating:

$$C_\delta = \left\{ x \in \mathbb{R}^3,\ 1 - \delta < |x| < 1 \right\}.$$

See Figure 1.13 for an illustration.

Assume that the map $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ is defined for $x_\Gamma \in \Gamma$ by:

$$\psi_\Gamma(x_\Gamma) := (\theta, \phi),$$

where (θ, ϕ) is the unique solution of of $x = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$.(See Figure 1.13) Figure 1.12(a) is an illustration when the reference function $(\hat{x}_1, \hat{x}_2, \hat{\nu}) \mapsto \hat{\rho}(\hat{x}_1, \hat{x}_2, \hat{\nu})$ only depends of the argument $\hat{x}_1$. Figure 1.12(a) is an illustration when the reference function $(\hat{x}_1, \hat{x}_2, \hat{\nu}) \mapsto \hat{\rho}(\hat{x}_1, \hat{x}_2, \hat{\nu})$ only depends of the argument $\hat{x}_2$. Finally Figure 1.12(a) is an illustration for general function $\hat{\rho}$.

1.2.4 The problem of fast cell contractions

Assume that Γ is the unit sphere and chose the map ψ_Γ as the polar coordinates (1.2.2) then we get Figure 1.2.4. Let us define the deformed cells as the images of the cells in the plane through the map ${\psi_\Gamma}^{-1}$. If we reduce the small parameter (see Figure 1.2.4) we graphically see that the deformed cells shrink faster around the poles of the sphere. This phenomena is not consistent with the intuitive idea of a periodic coating. Moreover it induces difficulties in the analysis. Let us explain more precisely this problem. The

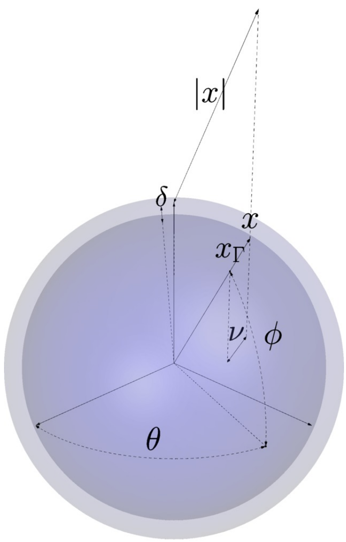


Figure 1.13: Illustration of the map $\mathcal{L}$ and the thin coat C_δ in the case of the unit sphere

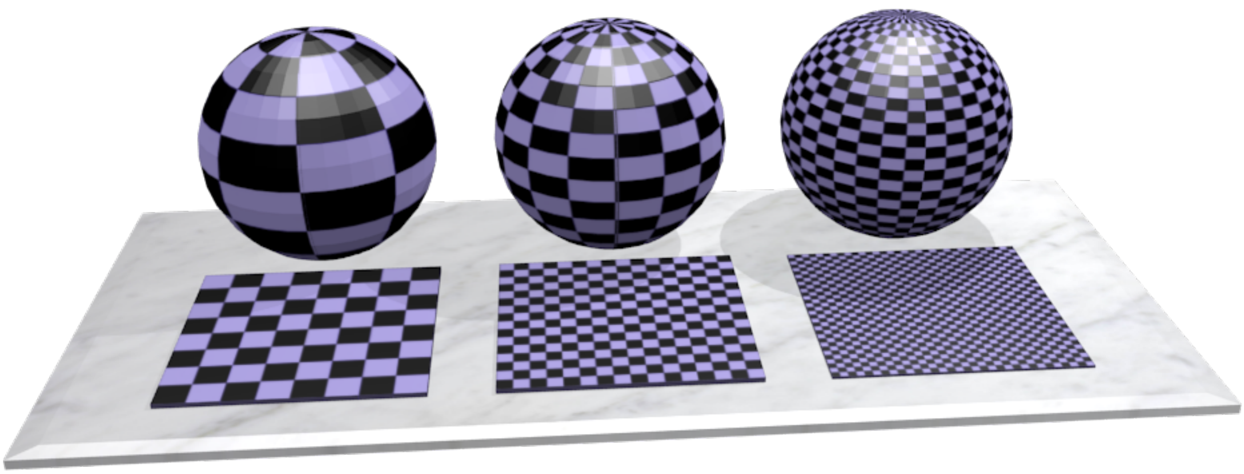


Figure 1.14: Problem of cell contraction

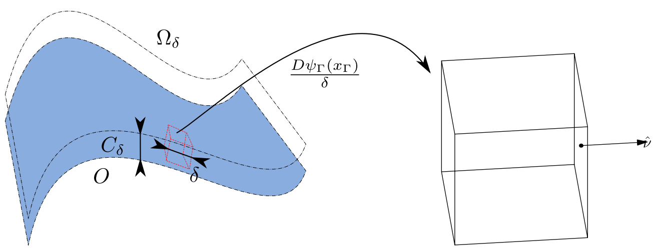


Figure 1.15: Transformation of a microscopic cell

microscopic cell on $x_\Gamma \in \Gamma$ denoted $\hat{Y}^{\delta}_{x_\Gamma}$ is mathematically defined by (see Figure 1.15):

$$\hat{Y}^{\delta}_{x_\Gamma} := \mathcal{L}^{-1}\Big({\psi_\Gamma}^{-1} \left(x_r +]0,\delta[^2\right) \times]-\delta, 0[\Big) \quad \text{with} \quad x_r := \psi_\Gamma(x_\Gamma).$$

For the sequel some elements from differential geometry are required. We assume that O is a $C^{m_\Gamma+1}$ domain for some $m_\Gamma + 1 \geq 2$ in the sense of [60, 2.5.2 Surfaces and Sobolev spaces]. We summarize in the following result the most useful properties of boundaries Γ with this type of regularity:

- There exists a collection of functions $\left(\phi_{x_\Gamma} : V_{x_\Gamma}(0) \subset \mathbb{R}^2 \mapsto W_{x_\Gamma}(x_{x_\Gamma})\right)_{x_\Gamma \in \Gamma}$ where $V_{x_\Gamma}(0)$ and $W_{x_\Gamma}(x_{x_\Gamma})$ are neighborhoods of respectively 0 in $\mathbb{R}^2$ and x_{x_Γ} in Γ in the sense of subspace topology of Γ. (We recall that the subspace topology of Γ is defined by $\{X \cap \Gamma,\ X \text{ is a open set of } \mathbb{R}^3\}$.)
- For all $x_\Gamma \in \Gamma$, if we see the function ϕ_{x_Γ} as a map $\phi_{x_\Gamma} : V_{x_\Gamma}(0) \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$ then this last function is of class $C^{m_\Gamma+1}$.
- For all $x_\Gamma \in \Gamma$ the application $\phi_{x_\Gamma} : V_{x_\Gamma}(0) \subset \mathbb{R}^2 \mapsto W_{x_\Gamma}(x_\Gamma)$ is bijective. In other term, one has a local system of coordinates around x_Γ for wich $\phi_{x_\Gamma}(0,0) = x_\Gamma$ and $\phi_{x_\Gamma}(u_1,u_2)$ is a point in the neighborhood of x_Γ, which coordinate are in $\mathbb{R}^3$ are:

  $$\begin{pmatrix} \phi^1_{x_\Gamma}(u_1,u_2) \\ \phi^2_{x_\Gamma}(u_1,u_2) \\ \phi^3_{x_\Gamma}(u_1,u_2) \end{pmatrix}.$$

  (See Figure 1.17).
- For all $x_\Gamma \in \Gamma$ the linear application $\mathrm{D}\,\phi_{x_\Gamma}(0) \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is injective where D is the classical differential operator defined for map:

  $$X : (u_1,u_2) \in V_{x_\Gamma}(0) \mapsto \big(X_1(u_1,u_2), X_2(u_1,u_2), X_3(u_1,u_2)\big),$$

  and $(u_1,u_2) \in V_{x_\Gamma}(0)$ by the following matrix:

  $$\mathrm{D}\,X(u_1,u_2) := \begin{pmatrix} \partial_{u_1} X_1(u_1,u_2) & \partial_{x_2} X_1(u_1,u_2) \\ \partial_{u_1} X_2(u_1,u_2) & \partial_{u_2} X_2(u_1,u_2) \\ \partial_{u_1} X_3(u_1,u_2) & \partial_{u_2} X_3(u_1,u_2) \end{pmatrix}.$$

- The tangent space of Γ at the point x_Γ is the following space:

  $$T_{x_\Gamma}\Gamma := \operatorname{Im} \mathrm{D}\,\phi_{x_\Gamma}(0).$$

  (See Figure 1.16). We emphasize that since by assumption $\mathrm{D}\,\phi_{x_\Gamma}(0)$ is an injective map then the space $T_{x_\Gamma}\Gamma$ is a $2-$dimensional and we have the basis $(e_1(x_\Gamma), e_2(x_\Gamma))$ where:

  $$e_1(x_\Gamma) := \begin{pmatrix} \partial_{u_1}\phi^1_{x_\Gamma}(0) \\ \partial_{u_1}\phi^2_{x_\Gamma}(0) \\ \partial_{u_1}\phi^3_{x_\Gamma}(0) \end{pmatrix} \quad \text{and} \quad e_2(x_\Gamma) \begin{pmatrix} \partial_{u_2} = \phi^1_{x_\Gamma}(0) \\ \partial_{u_2}\phi^2_{x_\Gamma}(0) \\ \partial_{u_2}\phi^3_{x_\Gamma}(0) \end{pmatrix}.$$

  The orthogonal of the space $T_{x_\Gamma}\Gamma$ is the the linear span of the vector $n(x_\Gamma)$:

  $$T_{x_\Gamma}\Gamma = n(x_\Gamma)^\perp$$

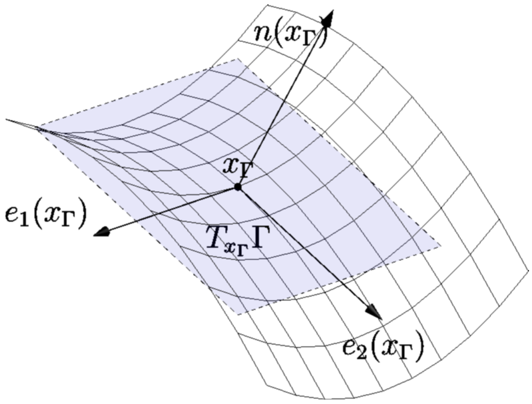


Figure 1.16: Illustration of the space $T_{x_\Gamma}\Gamma$

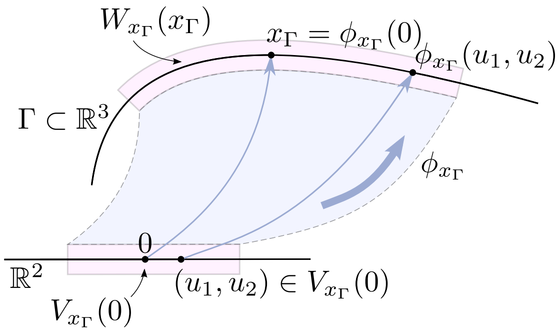


Figure 1.17: Illustration of manifold in the 2D case

- For all $(x_\Gamma, y_\Gamma) \in \Gamma^2$ if $W_{x_\Gamma}(x_\Gamma) \cap W_{y_\Gamma}(y_\Gamma) \neq \emptyset$ then the map:

$$\phi_{y_\Gamma}^{-1} \circ \phi_{x_\Gamma} : \phi_{x_\Gamma}^{-1}\big(W_{x_\Gamma}(x_\Gamma) \cap W_{y_\Gamma}(y_\Gamma)\big) \mapsto W_{y_\Gamma}(y_\Gamma),$$

  is of class $C^{m_\Gamma+1}$(See Figure 1.18).

We now give the defintion of differential for functions f defined on the surface Γ. Indeed we say that an application $f : \Gamma \mapsto \mathbb{R}^d$ is of class C^k with $d \in \mathbb{N}$ if for all $x_\Gamma \in \Gamma$ the application:

$$f \circ \phi_{x_\Gamma} : V_{x_\Gamma} \mapsto \mathbb{R}^d,$$

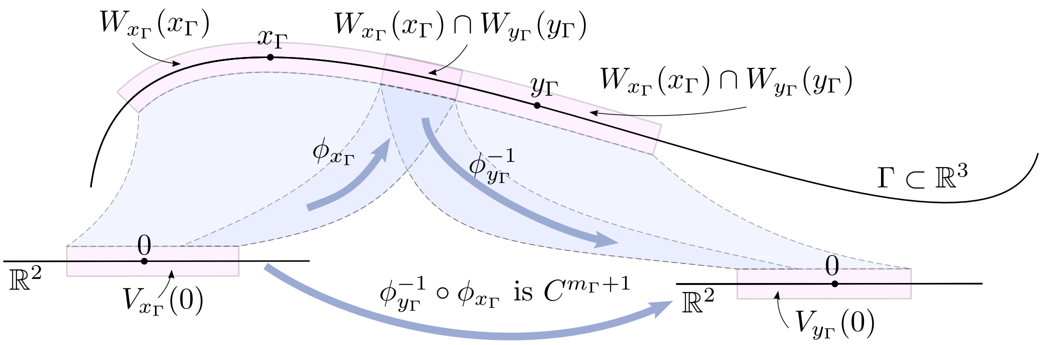


Figure 1.18: Illustration of the change of chart

is of class C^k. Under this last condition we define the differential symbol $\mathrm{D}\, f(x_\Gamma)$: $T_{x_\Gamma}\Gamma \mapsto \mathbb{R}^d$ by:

$$\mathrm{D}\, f(x_\Gamma) := \mathrm{D}(f \circ \phi_{x_\Gamma})(0) \cdot (\mathrm{D}\, \phi_{x_\Gamma}(0))^{-1}\,.$$

Since $\big(e_1(x_\Gamma), e_2(x_\Gamma)\big)$ is a basis of the space $T_{x_\Gamma}\Gamma$, this last definition is equivalent to define $\mathrm{D}\, f(x_\Gamma)$ as the unique linear operator on $T_{x_\Gamma}\Gamma$ such that:

$$\mathrm{D}\, f(x_\Gamma) \cdot \begin{pmatrix} \partial_{u_1}\phi^1_{x_\Gamma}(0) \\ \partial_{u_1}\phi^2_{x_\Gamma}(0) \\ \partial_{u_1}\phi^3_{x_\Gamma}(0) \end{pmatrix} = \partial_{u_1}(f \circ \phi_{x_\Gamma})(0) \quad \text{and} \quad \mathrm{D}\, f(x_\Gamma) \cdot \begin{pmatrix} \partial_{u_2}\phi^1_{x_\Gamma}(0) \\ \partial_{u_2}\phi^2_{x_\Gamma}(0) \\ \partial_{u_2}\phi^3_{x_\Gamma}(0) \end{pmatrix} = \partial_{u_2}(f \circ \phi_{x_\Gamma})(0).$$

Thus thanks to these definitions if ψ_Γ is differentiable and locally injective at the point $x_\Gamma \in \Gamma$, we can introduce the quantity $\mathcal{G} : \Gamma \mapsto \mathbb{R}$(called the determinant of the metric tensor associated with ψ_Γ) defined by

$$\mathcal{G}(x_\Gamma) := \det\Big(\mathrm{D}\, \psi_\Gamma(x_\Gamma)\, \mathrm{D}\, \psi^\dagger_\Gamma(x_\Gamma)\Big)^{-\frac{1}{2}}.$$

(See (1.2.14) for an example of this last quantity when Γ is the unit sphere.) For a Banach space F, we denote by $F^\dagger$ the dual of F and for an operator $A : E \mapsto F$ we denote by $A^\dagger$ the dual operator $F^\dagger \mapsto E^\dagger$ associated with the duality products $<,>_{F^\dagger - F}$ and $<,>_{E^\dagger - E}$. Thanks to the change of variable formula for integrals we get that if ψ_Γ is differentiable and locally injective at the point x_Γ then the Lebesgue-measure of $\hat{Y}^\delta_{x_\Gamma}$ satisfies :

$$\mu\left(\hat{Y}^\delta_{x_\Gamma}\right) \underset{\delta \to 0}{\sim} \mathcal{G}(x_\Gamma) \cdot \delta^3,$$

If the quantity $\mathcal{G}$ is not bounded from below and above by positive constants then this corresponds with the first problem we previously stated. We shall see that $\mathcal{G}$ is needed to be bounded from below by a positive constant in order to prove the convergence of the asymptotic expansions.

Proposition 1.2.6. *If one of the connected component of Γ is diffeomorphic to the unit sphere S^2 and ψ_Γ is C^1 then there exists at least one point $x^*_\Gamma \in \Gamma$ such that:*

$$\det\Big(\mathrm{D}\, \psi_\Gamma(x^*_\Gamma)\, \mathrm{D}\, \psi^\dagger_\Gamma(x^*_\Gamma)\Big) = 0.$$

The proof of this result is a direct application of the hairy ball theorem that we recall hereafter. First we introduce for x_Γ the tangent space of Γ at the point x_Γ denoted $T_{x_\Gamma}\Gamma$.

Definition 1.2.7. *Let $X : \Gamma \mapsto \mathbb{R}^3$. We say that X is a tangent vector field if for all $x_\Gamma \in \Gamma$ we have:*

$$X(x_\Gamma) \in T_{x_\Gamma}\Gamma.$$

Theorem 1.2.8 (Hairy ball theorem). *Let $n \in \mathbb{N} \setminus \{0\}$, $S^{2n} \subset \mathbb{R}^{2n+1}$ be the unit sphere and X be a continuous tangent vector field on S^{2n} then $X(x^*_\Gamma) = 0$ for some $x^*_\Gamma \in \Gamma$.*

Proof of Proposition 1.2.6. By hypothesis there exists a diffeomorphism from Γ into S^2 that we denoted by $\phi : \Gamma \mapsto S^2$. Then introduce the vector field X_1 on S^2 defined for $x_{S^2} \in S^2$ by:

$$X(x_{S^2})_1 := \mathrm{D}\, \phi(x_\Gamma)\, \mathrm{D}\, \psi_\Gamma(x_\Gamma)^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

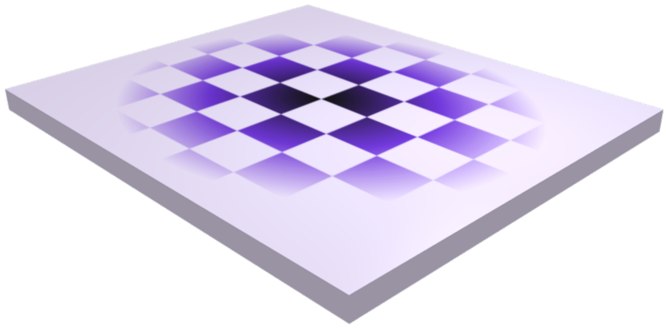


Figure 1.19: The plate for patching solution

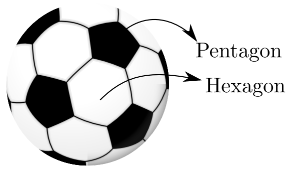


Figure 1.20: Paving with hexagon and pentagon of soccer balls

where $x_\Gamma := \phi^{-1}(x_{S^2})$. Thus we can apply the hairy ball theorem, we have that there exists $x_{S^2} \in S^2$ such that $X(x_{S^2})_1 = 0$. Define $x_\Gamma^* := \phi^{-1}(x_{S^2})$ then we have:

$$0 = \mathrm{D}\,\phi(x_\Gamma^*)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma^*)^\dagger \cdot (1,0)^\dagger,$$

and using that $\mathrm{D}\,\phi(x_\Gamma^*)$ is invertible yields $\left(\mathrm{D}\,\psi_\Gamma(x_\Gamma^*)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma^*)^\dagger\right) \cdot (1,0)^\dagger = 0$. Therefore we finished the proof.

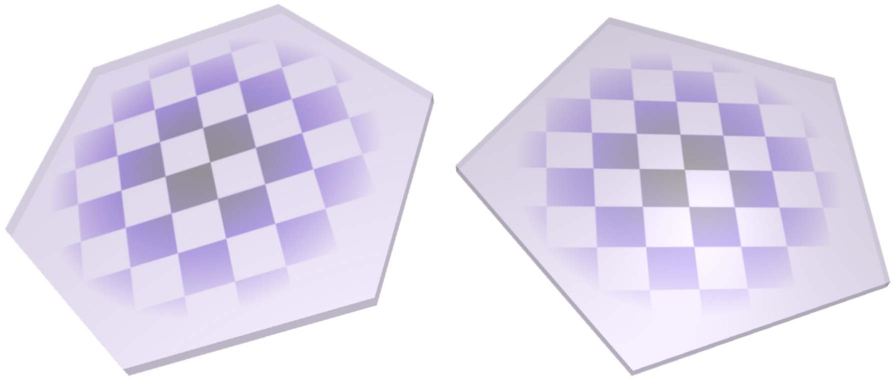
1.2.5 The patching solution

In order to avoid the problem of vanishing $\mathcal{G}$ we shall assume that the periodicity is defined on subsets of Γ that are glued together. An example of patching is Figure 1.19, Figure 1.21(a), Figure 1.21(b) and Figure 1.22 The matching process is be modeled by the way we construct the map ψ_Γ. Since the surface Γ is compact we can define $N_\Gamma \in \mathbb{N}$ by the smallest number such that there exists a family of points $\left(x_\Gamma^1, \cdots, x_\Gamma^{N_\Gamma}\right) \in \Gamma^{N_\Gamma}$ such that we have:

$$\Gamma = \bigcup_{i=0}^{N_\Gamma} W_{x_\Gamma^i}(x_\Gamma^i). \tag{1.2.5}$$

Next we introduce a smooth partition of unity $\left(\chi_i\right)_{1\le i\le N_\Gamma}$ associated to the open cover $\left(W_{x_\Gamma^i}(x_\Gamma^i)\right)_{1\le i\le N_\Gamma}$. Finally define our map $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ for $x_\Gamma \in \Gamma$ by:

$$\psi_\Gamma(x_\Gamma) := \sum_{i\in I(x_\Gamma)} \chi_i(x_\Gamma)\phi^{-1}_{x_\Gamma^i}(x_\Gamma) \quad \text{with} \quad I(x_\Gamma) := \left\{1 \le i \le N_\Gamma, x_\Gamma \in W_{x_\Gamma^i}(x_\Gamma^i)\right\}$$



(a) a hexagon

(b) a pentagon

Figure 1.21: Cut plate

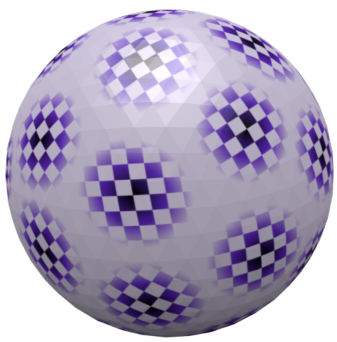


Figure 1.22: The $\psi_\Gamma - \delta -$ periodicity for soccer ball

We recall that the problem comes from the region where $\mathrm{D}\,\psi_\Gamma$ becomes singular, i.e. $\mathcal{G}$ vanishes. We introduce for arbitrary $\mathcal{G}_{\min} > 0$ and $\mathcal{G}_{\max}$ the following open set of non singular points:

$$\Gamma_{\mathrm{M}} := \Big\{ x_\Gamma \in \Gamma, \mathcal{G}(x_\Gamma) > \mathcal{G}_{\min} \text{ and } |\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)| < \mathcal{G}_{\max} \Big\}. \tag{1.2.6}$$

Proposition 1.2.9. *If $\mathcal{G}_{\min}$ is small enough then we have for all $1 \leq i \leq N_\Gamma$ that:*

$$\Gamma_{\mathrm{M}} \cap W_{x_\Gamma^i}(x_\Gamma^i) \neq \emptyset.$$

Proof. Let $1 \leq i \leq N_\Gamma$. We introduce the following set:

$$W_i := \left(\Gamma \setminus \bigcup_{j \neq i} W_{x_\Gamma^j}(x_\Gamma^j) \right),$$

and a sufficient condition to our stated result is to show that for some choice of $\mathcal{G}_{\min}$ we have:

$$W_i \neq \emptyset \quad \text{and} \quad W_i \subset \Gamma_{\mathrm{M}} \cap W_{x_\Gamma^i}(x_\Gamma^i). \tag{1.2.7}$$

Indeed if we have $W_i = \emptyset$ then we have:

$$\Gamma = \bigcup_{j \neq i} W_{x_\Gamma^j}(x_\Gamma^j),$$

which contradicts that N_Γ is the small number such that we have (1.2.5). Thanks to (1.2.5) we have:

$$W_i = \left(\bigcup_{j=1}^{N_\Gamma} W_{x_\Gamma^j}(x_\Gamma^j) \right) \setminus \bigcup_{j \neq i} W_{x_\Gamma^j}(x_\Gamma^j) = W_{x_\Gamma^i}(x_\Gamma^i) \setminus \bigcup_{j \neq i} W_{x_\Gamma^j}(x_\Gamma^j),$$

which leads to:

$$W_i \subset W_{x_\Gamma^j}(x_\Gamma^j). \tag{1.2.8}$$

Now let us prove that :

$$\inf_{x_\Gamma \in W_i} \mathcal{G}(x_\Gamma) > 0. \tag{1.2.9}$$

Indeed we have for all $x_\Gamma \in W_i$ that $\chi_i(x_\Gamma) = 1$ and $\chi_j(x_\Gamma) = 0$ for all $j \neq i$. Since 0 and 1 are the minimal and max values that for all j the function χ_j we also have $\mathrm{D}\,\chi_j(x_\Gamma) = 0$. Thus we have thanks to Leibniz formula that:

$$\mathrm{D}\,\psi_\Gamma(x_\Gamma) = \sum_{j \in I(x_\Gamma)} \mathrm{D}\left(\chi_i \phi_{x_\Gamma^j}^{-1}\right)(x_\Gamma) = \sum_{j \in I(x_\Gamma)} \left(\cancel{\phi_{x_\Gamma^j}^{-1} \mathrm{D}\,\chi_i} + \chi_i\, \mathrm{D}\,\phi_{x_\Gamma^j}^{-1} \right)(x_\Gamma) = \mathrm{D}\,\phi_{x_\Gamma^i}^{-1}(x_\Gamma^i),$$

which leads to :

$$\mathcal{G}(x_\Gamma) = \det\Big(\mathrm{D}\,\phi_{x_\Gamma^i}^{-1}(x_\Gamma)\, \mathrm{D}^\dagger\, \phi_{x_\Gamma^i}^{-1}(x_\Gamma) \Big). \tag{1.2.10}$$

Moreover we recall that $\phi_{x_\Gamma^i}^{-1} : W_{x_\Gamma^i}(x_\Gamma^i) \mapsto V_{x_\Gamma^i}(0)$ is a diffeomorphism and combining with the compactness of the set W_i yields:

$$\inf_{x_\Gamma \in W_i} \det\Big(\mathrm{D}\,\phi_{x_\Gamma^i}^{-1}(x_\Gamma)\, \mathrm{D}^\dagger\, \phi_{x_\Gamma^i}^{-1}(x_\Gamma) \Big).$$

Combining with(1.2.10), we conclude the proof of (1.2.9).

Now assume that:

$$0 < \mathcal{G}_{\min} < \inf_{x_\Gamma \in W_i} \mathcal{G}(x_\Gamma).$$

Thus we directly get that $W_i \subset \Gamma_{\mathrm{M}}$ and combining with (1.2.8) conclude the proof of (1.2.7) which in turn, concludes the whole proof. $\square$

Let $P(\hat{\Omega})$ be the set of functions $(\hat{x}, \hat{\nu}) \mapsto \tilde{u}(\hat{x}, \hat{\nu})$ defined on $\hat{\Omega}$ such that $\tilde{u}$ is one periodic in the variable $\hat{x}$ i.e.

$$\tilde{u}(\hat{x} + m, \hat{\nu}) = \tilde{u}(\hat{x}, \hat{\nu}), \ \forall(\hat{x}, \hat{\nu}, m) \in \hat{\Omega} \times \mathbb{Z}^2. \tag{1.2.11}$$

Definition 1.2.10. *Let $\hat{u} : \Gamma \mapsto P(\hat{\Omega})$ be a reference function we say that $\hat{u}$ is patching-ψ_Γ-admissible if for all $x_\Gamma \notin \Gamma_{\mathrm{M}}$ the function $\hat{u}(x_\Gamma)$ only depends on the argument $\hat{\nu}$ i.e.*

$$\forall x_\Gamma \notin \Gamma_{\mathrm{M}}, \ \exists \hat{u}_{\hat{\nu}}(x_\Gamma) :]-1, 0[\mapsto \mathbb{R} \quad st \quad \forall(\hat{x}, \hat{\nu}) \in \mathbb{R}^2 \times]-1, 0[\quad \hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) = \hat{u}_{\hat{\nu}}(x_\Gamma; \hat{\nu}).$$

Let $u \in P(\hat{\Omega})$ and χ be a smooth function vanishing on $\Gamma \setminus \Gamma_{\mathrm{M}}$ then the function $\hat{u}_\chi$ defined for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ by $\hat{u}_\chi(x_\Gamma; \hat{x}, \hat{\nu}) := \chi(x_\Gamma)\hat{u}(\hat{x}, \hat{\nu})$ is an example of a patching admissible function.

For the remainder of our work the initial definition of $\psi_\Gamma - \delta -$periodicity is now replaced by the following one:

Definition 1.2.11. *Let $(u^\delta)_{\delta>0}$ be a sequence of function defined on the thin coating C^δ. We say that $(u^\delta)_{\delta>0}$ is $\psi_\Gamma - \delta -$periodic if there exists a function $\hat{u} : \Gamma \mapsto P(\hat{\Omega})$ patching-ψ_Γ-admissible such that for all $\delta > 0$*

$$\forall x \in C^\delta, \ u^\delta(x) := \hat{u}\big(x_\Gamma; \hat{x}, \hat{\nu}\big) \ \textit{where} \quad (x_\Gamma, \nu) := \mathcal{L}(x) \ \textit{and}\, (\hat{x}, \hat{\nu}) := \frac{(\psi_\Gamma(x_\Gamma), \nu)}{\delta}.$$

The example of the sphere:

Now let us illustrate the exemple of the sphere, when ψ_Γ is the spherical coordinate system. This mean that the map $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ is defined for $x_\Gamma \in \Gamma$ by:

$$\psi_\Gamma(x_\Gamma) := (\theta, \phi),$$

where (θ, ϕ) is the unique solution of of $x = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$.(See Figure 1.13). Define:

$$\Gamma^* := \Gamma \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\},$$

and define the map $\phi_\Gamma : [0, 2\pi[\times] - \frac{\pi}{2}, \frac{\pi}{2}[\mapsto \Gamma^*$ for $(\theta, \phi) \in [0, 2\pi[\times] - \frac{\pi}{2}, \frac{\pi}{2}[$ by:

$$\phi_\Gamma \begin{pmatrix} \theta \\ \phi \end{pmatrix} := \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ \sin\phi \end{pmatrix}.$$

The map ϕ_Γ is a C^∞ function and its differential is given for θ, ϕ by:

$$\mathrm{D}\,\phi_\Gamma(\theta, \phi) = \begin{pmatrix} -\cos(\phi)\sin(\theta) & -\sin(\phi)\cos(\theta) \\ \cos(\phi)\cos(\theta) & -\sin(\phi)\sin(\theta) \\ 0 & \cos(\phi) \end{pmatrix}. \tag{1.2.12}$$

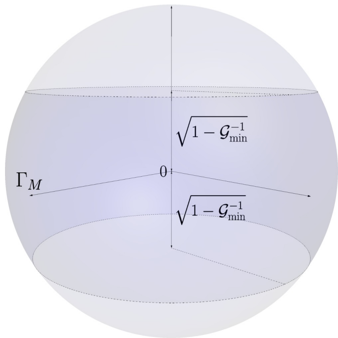


Figure 1.23: Illustration of the set Γ_{M} in the case a the unit sphere

From this we have that:

$$\mathrm{D}\,\phi_\Gamma(\theta,\phi)^\dagger\,\mathrm{D}\,\phi_\Gamma(\theta,\phi) = \begin{pmatrix} \cos^2(\theta) & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.2.13}$$

Thus from this, we can see that if $\theta \notin \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ then the matrix $\mathrm{D}\,\phi_\Gamma(\theta,\phi)^\dagger\,\mathrm{D}\,\phi_\Gamma(\theta,\phi)$ is injective where $x_\Gamma := \phi_\Gamma(\theta,\phi)$. Thus in this case the operator $\mathrm{D}\,\phi_\Gamma(\theta,\phi) : \mathbb{R}^2 \mapsto T_{x_\Gamma}\Gamma$ is bijective. Therefore one can deduce that the map $\phi_\Gamma : [0,2\pi[\times]-\frac{\pi}{2},\frac{\pi}{2}[\mapsto \Gamma^*$ is a C^∞ diffeomorphism.

Therefore the map $\psi_\Gamma : \Gamma^* \mapsto [0,2\pi[\times]-\frac{\pi}{2},\frac{\pi}{2}[$ is a diffeomorphism and we have for all $x_\Gamma \in \Gamma^*$ that:

$$\mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger = \big(\mathrm{D}\,\phi_\Gamma(\theta,\phi)^\dagger\,\mathrm{D}\,\phi_\Gamma(\theta,\phi)\big)^{-1} = \begin{pmatrix} \cos^{-2}(\theta) & 0 \\ 0 & 1 \end{pmatrix},$$

where $(\theta,\phi) := \psi_\Gamma(x_\Gamma)$. Thus we have:

$$\mathcal{G}(x_\Gamma) = \cos^{-2}(\theta), \tag{1.2.14}$$

and then according to the definition (1.2.6) of the set Γ_{M}, one has:

$$\Gamma_{\mathrm{M}} = \{(\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi),\ \theta \in [0,2\pi[\text{ and } -\eta < \phi < \eta\}, \tag{1.2.15}$$

where $\eta > 0$ is a the unique solution in $[0,\frac{\pi}{2}[$ of $\cos^{-2}(\eta) = \mathcal{G}_{\max}$. Here we have chosen $\mathcal{G}_{\min} = 1$. One can easily check that we can rewrite the set Γ_{M} as follow:

$$\Gamma_{\mathrm{M}} = \left\{(x,y,z) \in \Gamma, -\sqrt{1-\mathcal{G}_{\max}^{-1}} < z < \sqrt{1-\mathcal{G}_{\max}^{-1}}\right\}.$$

See Figure 1.23 for a graphical illustration of this last set. Finally, Figure 1.24 is a graphical illustration of a $\psi_\Gamma - \delta -$ periodic function associated to a patching admissible reference function.

1.3 Reformulation of the Helmholtz equation in the surface local coordinates

We recall the problem that we are interested in: Find $u^\delta \in H^1_{\mathrm{loc}}(\Omega^\delta)$ such that:

$$\begin{cases} \operatorname{div}\big(\rho^\delta\,\nabla u^\delta\big) + k^2\mu^\delta\,u^\delta = f, & \text{in } \Omega^\delta, \\ \partial_{n^\delta} u_\delta = 0 & \text{on } \partial\Omega^\delta, \end{cases}$$

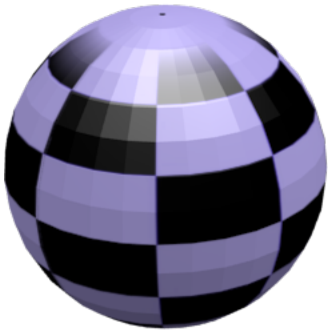


Figure 1.24: Illustration of the $\psi_\Gamma - \delta -$periodicity in the case of the unit sphere.

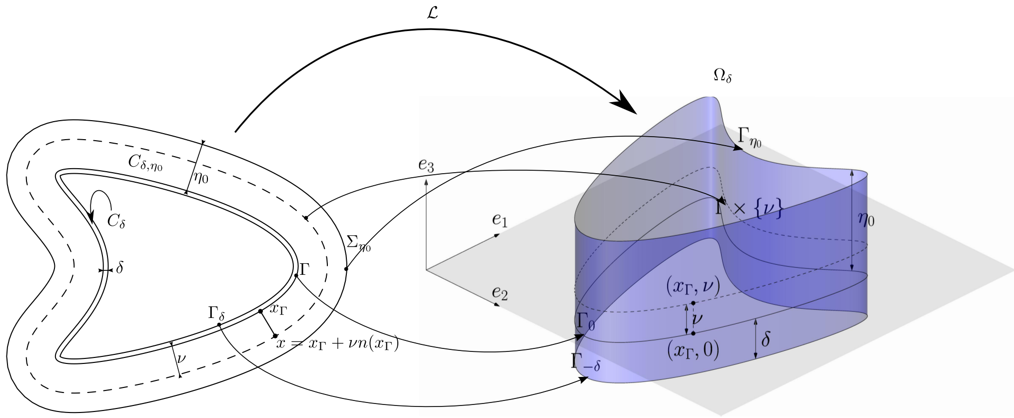


Figure 1.25: $\mathcal{L} : C_{\delta,\eta_0} \mapsto \Gamma\times]-\delta, \eta_0[$

and u^δ satisfies the Sommerfeld radiation condition:

$$\lim_{R\to\infty} \int_{|x|=R} |\partial_r u^\delta - iku^\delta|^2 = 0.$$

We remark that the dependence of our geometry with respect to the small parameter δ does not seems trivial. However from the definition of the mapping $\mathcal{L}$ we directly get the following characterization of our boundary:

$$\partial\Omega^\delta = \mathcal{L}^{-1}(\Gamma \times \{-\delta\}) \quad \text{and} \quad \Gamma = \mathcal{L}^{-1}(\Gamma \times \{0\}).$$

Moreover from [19, **2.7.** Normal Bundles and Tubular Neighborhoods], we get the following result:

Proposition 1.3.1. *If Γ have C^n regularity for some $n \in \mathbb{N}$ then for all $0 < \delta < \eta_0$, the application $\mathcal{L} : C_{\delta,\eta_0} \mapsto \Gamma\times]-\delta, \eta_0[$ is a C^{n-1} diffeomorphism and its inverse is given for $(x_\Gamma, \nu) \in \Gamma\times]-\delta, \eta_0[$ by (see Figure 1.9):*

$$\mathcal{L}^{-1}(x_\Gamma, \nu) = x_\Gamma + \nu n(x_\Gamma).$$

Thus the local coordinates seem to be a better coordinate system to describe the thin coating. Indeed the dependence with respect to the small parameter δ of the set $\Gamma\times]-\delta, 0[$ is more explicit than the one of the thin coating $C_\delta = \{x \in O,\ \operatorname{dist}(x, \Gamma) < \delta\}$.

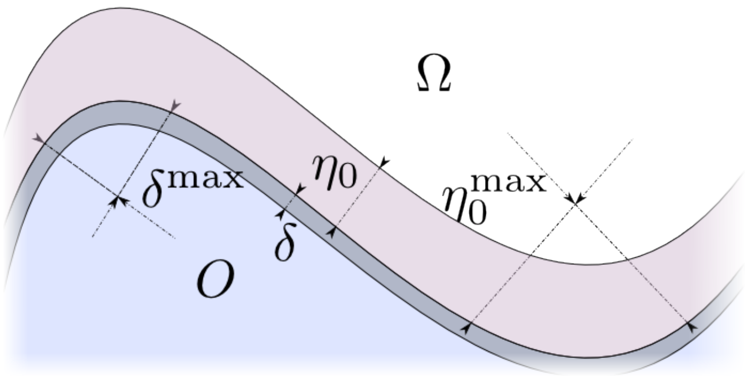


Figure 1.26: Illustration of the maximum value of δ and η_0

Nevertheless our problem is posed on the whole exterior Ω^δ and the application $\mathcal{L}$ is only defined on C^δ. According to Proposition 1.3.1, we assume that $\delta < \eta_0$ and the application $\mathcal{L}$ is now defined on the the set (See Figure 1.25)

$$C_{\delta,\eta_0} := \left\{x \in \overline{\Omega},\ \operatorname{dist}(x,\Gamma) < \eta_0\right\} \cup C^\delta.$$

We also assume that $\operatorname{supp}(f) \subset \mathbb{R}^3 \setminus C_{\delta,\eta_0}$ which is possible for η_0 small enough because we assumed that $\operatorname{supp}(f) \cap \Gamma = \emptyset$.

Let us explain why the expression of the inverse $\mathcal{L}^{-1}$ is more practical than the one of the map $\mathcal{L}$ for the sequel: Indeed, we recall that the map $\mathcal{L} : C_{\delta,\eta_0} \mapsto \Gamma \mapsto]-\delta,\eta_0[$ is given for $x \in C_{\delta,\eta_0}$ by $\mathcal{L}(x) := (x_\Gamma, \nu)$ where x_Γ is the unique minimizer of the functional:

$$x_\Gamma \in \Gamma \mapsto |x - x_\Gamma|,$$

and:

$$\nu := \operatorname{dist}(x,\Gamma) = \inf_{x_\Gamma\in\Gamma} |x - x_\Gamma| \text{ if } x_\Gamma \in \Omega \text{ and } \nu := -\operatorname{dist}(x,\Gamma) = -\inf_{x_\Gamma\in\Gamma} |x - x_\Gamma| \text{ if } x \notin \Omega.$$

This last definition is not explicit because we directly see that wee need to solve a problem of optimization:

$$\inf_{x_\Gamma\in\Gamma} |x - x_\Gamma|.$$

Therefore we cannot yet compute the derivative of the map $\mathcal{L}$ and we will this in the sequel that we will need to compute the quantity $\mathrm{D}\,\mathcal{L}$(See Proposition 1.3.3 for example). The expression of $\mathcal{L}^{-1}(x_\Gamma,\nu) = x_\Gamma + \nu n(x_\Gamma)$ is more explicit because it only requires to compute for $x_\Gamma \in \Gamma$ the normal unit vector $n(x_\Gamma)$. Moreover thanks to the definition of the tensor curvature $R(x_\Gamma) = \mathrm{D}\, n(x_\Gamma)$, we can easily establish that the differential of the inverse $\mathcal{L}^{-1}$ is given for $(x_\Gamma,\nu) \in \Gamma\times]-\delta,\eta_0[$ by the only linear operator on $T_{x_\Gamma}\Gamma \times \mathbb{R}$ defined for $(v_\Gamma, v_\nu) \in T_{x_\Gamma}\Gamma \times \mathbb{R}$ by:

$$\mathrm{D}\,\mathcal{L}^{-1}(x_\Gamma,\nu)\cdot\begin{pmatrix} u_\Gamma \\ u_\nu \end{pmatrix} = \big(\mathbb{I} + \nu R(x_\Gamma)\big)v_\Gamma + n(x_\Gamma)v_\nu. \tag{1.3.16}$$

Therefore we chose to derive a partial differential equation posed on the domain $\Gamma\times]-\delta,\eta_0[$ where the new unknown

$$u_\delta := u^\delta \circ \mathcal{L}^{-1}$$

$$
\begin{tikzcd}
x \in C_{\delta,\eta} \arrow[r, "\mathcal{L}"] \arrow[dr, "A^\delta"'] & (x_\Gamma, \nu) \in \Gamma \times]-\delta, \eta_0[\arrow[d, "A_\delta"] \\
& A^\delta(x) = A_\delta(x_\Gamma)
\end{tikzcd}
\qquad
\begin{tikzcd}
(x_\Gamma, \nu) \in \Gamma \times]-\delta, \eta_0[\arrow[r, "\mathcal{L}^{-1}"] \arrow[dr, "A_\delta"'] & x \in C_{\delta,\eta} \arrow[d, "A^\delta"] \\
& A^\delta(x) = A_\delta(x_\Gamma)
\end{tikzcd}
$$

Figure 1.27: The maps A^δ and A_δ

will be the solution. We emphasize our notation convention: function with upper-script δ are defined on $C_{\delta,\eta}$ while function with subscripts δ are defined on $\Gamma \times]-\delta, \eta_0[$.

Moreover, if A^δ is a function defined on C_{δ,η_0} then A_δ is defined by:

$$A_\delta = A^\delta \circ \mathcal{L}^{-1}.$$

Before obtaining a partial differential equation satisfied by the function u_δ we first need to reduce our problem (1.1.1) on the bounded domain C_{δ,η_0}.

1.3.1 Reduction of the exterior problem to the bounded domain C_{δ,η_0}

We use a classical way of reduction to a bounded domain through the Dirichlet to Neumann operator (See [43, 60]). Nevertheless a difficulty is that the support of the source term f might not be included into C_{δ,η_0}. To solve this problem we introduce an auxiliary function $u_f : \Omega \setminus C_{\delta,\eta_0} \mapsto \mathbb{C}$ defined as the unique solution of: Find $u_f \in H^1_{\mathrm{loc}}(\Omega \setminus C_{\delta,\eta_0})$ such that :

$$\begin{cases} \Delta u_f + k^2 u_f = f, & \text{in } \Omega \setminus C_{\delta,\eta_0}, \\ u_f = 0, & \text{on } \Sigma_{\eta_0} \end{cases}$$

and u_f satisfies the Sommerfeld radiation condition, where (see Figure 1.25):

$$\Sigma_{\eta_0} := \{x \in \Omega, \operatorname{dist}(x, \Gamma) = \eta_0\}.$$

The Dirichlet to Neumann map on Σ_{η_0}:

$$\mathrm{DtN} : H^{\frac{1}{2}}(\Sigma_{\eta_0}) \mapsto H^{-\frac{1}{2}}(\Sigma_{\eta_0}),$$

is defined for $g \in H^{\frac{1}{2}}(\Sigma_{\eta_0})$ by $\mathrm{DtN}\, g := \partial_\nu u_g$ where u_g is the unique solution of: Find $u_g \in H^1_{\mathrm{loc}}(\Omega \setminus C_{\delta,\eta_0})$ such that :

$$\begin{cases} \Delta u_g + k^2 u_g = 0, & \text{in } \Omega \setminus C_{\delta,\eta_0}, \\ u_g = g & \text{on } \Sigma_{\eta_0} \end{cases} \tag{1.3.17}$$

and u_g satisfies the Sommerfeld radiation condition. Finally we define:

$$\langle \cdot, \cdot \rangle_{\Sigma_{\eta_0}} := \langle \cdot, \cdot \rangle_{H^{-\frac{1}{2}}(\Sigma_{\eta_0}) - H^{\frac{1}{2}}(\Sigma_{\eta_0})}.$$

Let us explain why this last operator is well defined. Indeed, when the boundary Σ_{η_0} is at least Lipschitz, then according to the scattering theory, the problem (1.3.17) is well

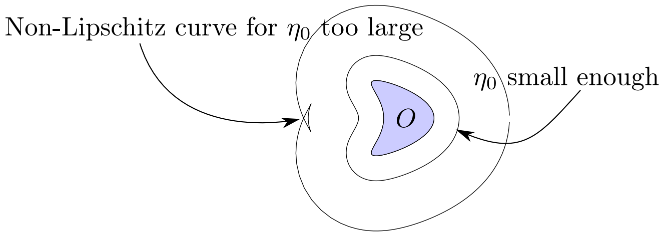


Figure 1.28: Regularity of Σ_{η_0} with respect to η_0

posed([60]). Moreover, according to Proposition 1.3.1 if η_0 is small then Σ_{η_0} is a Lipschitz surface. (See Figure 1.28 for a graphical illustration and a graphical illustration about what appends when η_0 is too large). Therefore if we choose η_0 small enough then the Dirichlet to Neumann map is well defined. Then thanks to these two definitions, we can state the following result:

Proposition 1.3.2. *If* $\operatorname{supp}(f) \subset \mathbb{R}^3 \setminus C_{\delta,\eta_0}$ *then the function* u^δ *is the unique solution of : Find* $u^\delta \in H^1(C_{\delta,\eta_0})$ *such that for all* $v^\delta \in H^1(C_{\delta,\eta_0})$*:*

$$a^\delta(u^\delta, v^\delta) = \left\langle \partial_\nu u_f - \mathrm{DtN}\, u_f, v^\delta \right\rangle_{\Sigma_{\eta_0}}, \tag{1.3.18}$$

where a^δ *is the sesquilinear form defined for* (u^δ, v^δ) *by:*

$$a^\delta(u^\delta, v^\delta) := \int_{C_{\delta,\eta_0}} \rho^\delta \nabla u^\delta \cdot \nabla v^\delta - \mu^\delta k^2 u^\delta \cdot v^\delta + \left\langle \mathrm{DtN}\, u^\delta, v^\delta \right\rangle_{\Sigma_{\eta_0}}.$$

Proof. From the definition of the function u_f we directly get that the function $u - u_f$ satisfies $\Delta(u-u_f) + k^2(u-u_f) = 0$ and the Sommerfeld radiation condition. Thus $u - u_f$ satisfies on the boundary Σ_{η_0}:

$$\partial_\nu(u - u_f) = \mathrm{DtN}(u - u_f).$$

and then u satisfies on Σ_{η_0} the boundary condition $\partial_\nu u - \mathrm{DtN}\, u = \partial_\nu u_f - \mathrm{DtN}\, u_f$. Thus the proof is finished. □

1.3.2 The problem posed in $\Gamma \times]-\delta, \eta_0[$

1.3.2.1 Variational formulation

Here we will transform (1.3.18) into a variational formulation in the volume $\Omega_\delta := \Gamma \times]-\delta, \eta_0[$. Let us summarize how we proceed to rewrite our problem:

1. We describe a new sesquilinear form $a_\delta : H^1(\Omega_\delta) \times H^1(\Omega_\delta) \mapsto \mathbb{C}$ such that for all functions u_δ, u^δ, v_δ and v^δ we have :

   $$u^\delta = u_\delta \circ \mathcal{L} \quad \text{and} \quad v^\delta = v_\delta \circ \mathcal{L} \quad \Longrightarrow \quad a_\delta(u_\delta, v_\delta) = a^\delta(u^\delta, v^\delta). \tag{1.3.19}$$

   To do it we proceed with the following steps:

(a) We prove that for all functions u_δ and u^δ linked by $u^\delta = u_\delta \circ \mathcal{L}$ the following equivalence holds:

$$u_\delta \in H^1(\Omega_\delta) \Longleftrightarrow u^\delta \in H^1(C_{\delta,\eta}), \tag{1.3.20}$$

and there exists $C > 0$ that does not depend on u_δ and δ such that the following equivalence of norms property holds:

$$C^{-1} \cdot \|u^\delta\|_{H^1(C_{\delta,\eta_0})} \leq \|u_\delta\|_{H^1(\Omega_\delta)} \leq C \cdot \|u^\delta\|_{H^1(C_{\delta,\eta_0})}. \tag{1.3.21}$$

(b) We use the expression of the gradient in local coordinates

(c) We use the change of variables $\mathcal{L}$ for integrals.

2. We seek an element $f_{\Sigma_{\eta_0}} \in H^{-\frac{1}{2}}\left(\Gamma \times \{\eta_0\}\right)$ such that for v_δ and v^δ:

$$v^\delta = v_\delta \circ \mathcal{L} \quad \Longrightarrow \quad \langle f_{\Sigma_{\eta_0}}, v_\delta \rangle_{\Gamma\times\{\eta_0\}} = \left\langle \partial_\nu u_f - \mathrm{DtN}\, u_f, v^\delta \right\rangle_{\Gamma\times\{\eta_0\}},$$

where we defined $\langle\cdot,\cdot\rangle_{\Gamma\times\{\eta_0\}} := \langle\cdot,\cdot\rangle_{H^{-\frac{1}{2}}(\Gamma\times\{\eta_0\})-H^{\frac{1}{2}}(\Gamma\times\{\eta_0\})}$. We emphasize that the function $f_{\Sigma_{\eta_0}}$ depends on the choice of η_0 and on the right hand-side f. This function is a distribution on Γ_{η_0} which itself depend of η_0.

3. We deduce that (1.3.18) is equivalent to: Find $u_\delta \in H^1(\Omega_\delta)$ such that for all $v_\delta \in H^1(\Omega_\delta)$ we have:

$$a_\delta(u_\delta, v_\delta) = \langle f_{\Sigma_{\eta_0}}, v_\delta \rangle_{\Gamma\times\{\eta_0\}}.$$

To state the results for parts (a) and (b) some elements are needed:

- We extend the unit outward normal application $n : \Gamma \mapsto \mathbb{R}^3$ for x satisfyaing:

$$\mathrm{dist}(x, \Gamma) < \eta_0,$$

(η_0 is the quantity appearing in Proposition 1.3.1) which takes the form $x = x_\Gamma + \nu n(x_\Gamma)$ by $\tilde{n}(x) := n(x_\Gamma)$(see Proposition 1.3.1 for existence and uniqueness of (x_Γ, ν)). Thanks to Proposition 1.3.1, since we assumed that Γ is at least C^2, this last extension $\tilde{n}$ is a C^1 function. Therefore thanks to this last regularity property, we now can define the tensor curvature R and the mean curvature H for $x_\Gamma \in$ by :

$$R(x_\Gamma) := \mathrm{D}\,\tilde{n}(0) \quad \text{with} \quad \mathrm{D}\,\tilde{n} := \begin{pmatrix} \partial_{x_1} n_1 & \partial_{x_2} n_1 & \partial_{x_3} n_1 \\ \partial_{x_1} n_2 & \partial_{x_2} n_2 & \partial_{x_3} n_2 \\ \partial_{x_1} n_3 & \partial_{x_2} n_3 & \partial_{x_3} n_3 \end{pmatrix} \tag{1.3.22}$$

and $H(x_\Gamma) := \frac{\mathrm{tr}\left(R(x_\Gamma)\right)}{2}$. A second definition of the tensor $R(x_\Gamma)$ is the following one: Let $\left(\phi^1_{x_\Gamma}, \phi^2_{x_\Gamma}, \phi^3_{x_\Gamma}\right) : V_{x_\Gamma}(0) \subset \mathbb{R}^2 \mapsto W_{x_\Gamma}(x_\Gamma) \subset \Gamma$ be a chart. Then define the map N for $(u_1, u_2) \in V_{x_\Gamma}(0)$ by:

$$N(u_1, u_2) := n\left(\phi^1_{x_\Gamma}(u_1, u_2), \phi^2_{x_\Gamma}(u_1, u_2), \phi^3_{x_\Gamma}(u_1, u_2)\right).$$

We recall that:

$$T_{x_\Gamma}\Gamma = \mathrm{Vect}\left\{ \begin{pmatrix} \partial_{x_1}\phi^1_{x_\Gamma}(0) \\ \partial_{x_1}\phi^2_{x_\Gamma}(0) \\ \partial_{x_1}\phi^3_{x_\Gamma}(0) \end{pmatrix}, \begin{pmatrix} \partial_{x_2}\phi^1_{x_\Gamma}(0) \\ \partial_{x_2}\phi^2_{x_\Gamma}(0) \\ \partial_{x_2}\phi^3_{x_\Gamma}(0) \end{pmatrix} \right\} \tag{1.3.23}$$

and $T_{x_\Gamma}\Gamma^\perp = \mathrm{Vect}\{n(x_\Gamma)\}$. Thus the tensor $R(x_\Gamma)$ is defined as the unique linear operator on $\mathbb{R}^3$ such that:

$$R(x_\Gamma)\begin{pmatrix}\partial_{x_1}\phi^1_{x_\Gamma}(0)\\ \partial_{x_1}\phi^2_{x_\Gamma}(0)\\ \partial_{x_1}\phi^3_{x_\Gamma}(0)\end{pmatrix} := \partial_{x_1}N(0,0), \quad R(x_\Gamma)\begin{pmatrix}\partial_{x_2}\phi^1_{x_\Gamma}(0)\\ \partial_{x_2}\phi^2_{x_\Gamma}(0)\\ \partial_{x_2}\phi^3_{x_\Gamma}(0)\end{pmatrix} := \partial_{x_2}N(0,0)$$

and $R(x_\Gamma)n(x_\Gamma) = 0$.

We recall that for all $x_\Gamma \in \Gamma$ we have $\mathrm{Im}\,(R(x_\Gamma)) \subset T_{x_\Gamma}\Gamma$ and $R(x_\Gamma) : T_{x_\Gamma}\Gamma \mapsto T_{x_\Gamma}\Gamma$ is a symmetric tensor.

- We define the surface ∇_Γ operator for a function $u \in H^1(\Gamma)$ as follows: We extend the function u for x satisfying $\mathrm{dist}(x,\Gamma) < \eta_0$ by $\tilde{u}(x) := u(x_\Gamma)$ where (x_Γ, ν) is the unique solution of $x = x_\Gamma + n(x_\Gamma)\nu$. Thanks to Proposition 1.3.1, since we assumed that Γ is at least C^2, this last extension $\tilde{u}$ is a C^1 function. Then for $x_\Gamma \in \Gamma$, we now can define the surface gradient by $\nabla_\Gamma u(x_\Gamma) := \nabla\tilde{u}(x_\Gamma)$. A second definition of the surface gradient is the following one: We define the function U for $(u_1, u_2) \in V_{x_\Gamma}(0)$ by:
$$U(u_1,u_2) := u\big(\phi^1_{x_\Gamma}(u_1,u_2), \phi^2_{x_\Gamma}(u_1,u_2), \phi^3_{x_\Gamma}(u_1,u_2)\big).$$
Thus thanks to (1.3.23), the surface gradient $\nabla_\Gamma u(x_\Gamma)$ can be defined as the unique element of $T_{x_\Gamma}\Gamma$ such that
$$\nabla_\Gamma u(x_\Gamma)\begin{pmatrix}\partial_{x_1}\phi^1_{x_\Gamma}(0)\\ \partial_{x_1}\phi^2_{x_\Gamma}(0)\\ \partial_{x_1}\phi^3_{x_\Gamma}(0)\end{pmatrix} := \partial_{x_1}U(0,0), \quad \nabla_\Gamma u(x_\Gamma)\begin{pmatrix}\partial_{x_2}\phi^1_{x_\Gamma}(0)\\ \partial_{x_2}\phi^2_{x_\Gamma}(0)\\ \partial_{x_2}\phi^3_{x_\Gamma}(0)\end{pmatrix} := \partial_{x_2}U(0,0).$$

- Thus we can introduce the operator $\nabla_{\mathcal{L}}$ defined for $u_\delta \in H^1(\Omega_\delta)$ by:
$$\nabla_{\mathcal{L}}\, u_\delta := \nabla_\Gamma\, u_\delta + n\partial_\nu u_\delta.$$

Thanks to these definitions and gradient formula used in [14, Some notations and recalls of differential geometry] and [57, Theorem 3.23], and the expression of differencial of the map $\mathcal{L}$ given by (1.3.16), we can easy establish the following result:

Proposition 1.3.3. *The equivalence* (1.3.20) *and* (1.3.21) *holds true. Moreover for all* $u_\delta \in H^1(\Omega_\delta)$ *the following expression for gradient of* $u^\delta := u_\delta \circ \mathcal{L}$ *holds:*
$$\nabla u^\delta = \big((\mathbb{I} + \nu R)^{-1} \cdot \nabla_{\mathcal{L}}\, u_\delta\big) \circ \mathcal{L},$$
where $\mathbb{I}$ *is the identity matrix of* $\mathbb{R}^3$*.*

The change of variable formula

We introduce for convenience the function $C : \Gamma \mapsto \mathbb{R}$ for $(x_\Gamma, \nu) \in \Omega_\delta$ by:
$$C(x_\Gamma,\nu) := \det(\mathbb{I} + \nu R(x_\Gamma)) = 1 + 2\nu H(x_\Gamma) + \nu^2 G(x_\Gamma), \tag{1.3.24}$$
where $G(x_\Gamma) := \det(R(x_\Gamma))$ is the Gaussian curvature and $H(x_\Gamma) := tr\big(R(x_\Gamma)\big)/2$ is the mean curvature. In order to illustrate these last quantity, let us introduce the

principal curvature κ_1 and κ_2 which are defined as the eigenvalues of the operator $R(x_\Gamma) : T_{x_\Gamma}\Gamma \mapsto T_{x_\Gamma}\Gamma$(We recall $R(x_\Gamma)$ is a endomorphism of $T_{x_\Gamma}\Gamma$). In other word, there exists a isomorphism $P(x_\Gamma) : T_{x_\Gamma}\Gamma \mapsto \mathbb{R}^2$ such that:

$$R(x_\Gamma) = P(x_\Gamma)\begin{pmatrix}\kappa_1 & 0\\ 0 & \kappa_2\end{pmatrix}P(x_\Gamma)^{-1}.$$

Then by using this notation, we can rewrite the Gaussian curvature and the mean curvature as follows:

$$G(x_\Gamma) = \kappa_1\kappa_2 \quad \text{and} \quad H(x_\Gamma) = \frac{\kappa_1+\kappa_2}{2},$$

and rewrite (1.3.24) as follows:

$$C(x_\Gamma,\nu) := (1+\nu\kappa_1)(1+\nu\kappa_2) \quad \text{or} \quad C(x_\Gamma,\nu) := 1+\nu(\kappa_1+\kappa_2)+\kappa_1\kappa_2\nu^2.$$

We choose an arbitrary η_0 satisfying:

$$\eta_0 < \min\Big(\text{dist}\left(\text{supp}(f),\Gamma\right), \sqrt{G^2-H^2}-H\Big)$$

in order to get that that the matrix field $(x_\Gamma,\nu) \mapsto \mathbb{I}+\nu R(x_\Gamma)$ is uniformly definite-positive on Γ. This mean that we have existence of $C' > 0$ such that for all $x_\Gamma \in \Gamma$ and $p \in \mathbb{R}^3$ we have:

$$((\mathbb{I}+\nu R(x_\Gamma))\cdot p, p) \geq C'|p|^2.$$

Under this last condition we get $C \geq (C')^2$.

Thanks to this last definition and [14, Theorem 3.23] we can state the following result:

Proposition 1.3.4. *Let A_δ be in $L^1(\Omega_\delta)$ then $A^\delta := A_\delta \circ \mathcal{L} \in L^1(C_{\delta,\eta_0})$ and we have:*

$$\int_{C_{\delta,\eta_0}} A^\delta dx = \int_{\Omega_\delta} A_\delta C d\Gamma d\nu.$$

The Dirichlet to Neumann map $\text{DtN}_\mathcal{L}$

As a direct consequence of Proposition 1.3.3 the following sesquilinear form:

$$(\tilde{u}_\delta,\tilde{v}_\delta) \mapsto \langle \text{DtN}\,\tilde{u}_\delta\circ\mathcal{L}, \tilde{v}_\delta\circ\mathcal{L}\rangle_{\Sigma_{\eta_0}},$$

is continuous on $H^{\frac{1}{2}}\left(\Sigma_{\eta_0}\right)^2$. Therefore there exists an operator $\text{DtN}_\mathcal{L} : H^{\frac{1}{2}}(\Gamma\times\{\eta_0\}) \mapsto H^{-\frac{1}{2}}(\Gamma\times\{\eta_0\})$ such that for all $\tilde{u}_\delta,\tilde{v}_\delta \in H^{\frac{1}{2}}(\Gamma\times\{\eta_0\})$ we have:

$$\langle \text{DtN}_\mathcal{L}\,\tilde{u}_\delta,\tilde{v}_\delta\rangle_{\Gamma\times\{\eta_0\}} := \langle \text{DtN}\,\tilde{u}_\delta\circ\mathcal{L}, \tilde{v}_\delta\circ\mathcal{L}\rangle_{\Sigma_{\eta_0}}. \tag{1.3.25}$$

The sesquilinear form a_δ The sesquilinear form a_δ is defined for $(u_\delta,v_\delta) \in H^1(\Omega_\delta)^2$ by:

$$a_\delta(u_\delta,v_\delta) := \int_{\Omega_\delta} \rho_\delta\,\mathcal{C}\,\nabla_\mathcal{L} u_\delta\cdot\nabla_\mathcal{L} v_\delta - \mu_\delta\,\mathrm{C}\,u_\delta\cdot v_\delta + \langle \text{DtN}_\mathcal{L}\,u_\delta, v_\delta\rangle_{\Gamma\times\{\eta_0\}},$$

where the linear operator $\mathcal{C} : \Gamma\times\mathbb{R} \mapsto \mathcal{L}(\mathbb{R}^3)$ is defined for $(x_\Gamma,\nu) \in \Omega_\delta$ by the only linear operator such that for all $v_\Gamma \in T_{x_\Gamma}\Gamma$

$$\begin{cases} \mathcal{C}(x_\Gamma,\nu)\cdot v_\Gamma := \mathrm{C}(x_\Gamma,\nu)\cdot\left(\mathbb{I}+\nu R(x_\Gamma)\right)^{-2}\cdot v_\Gamma,\\ \mathcal{C}(x_\Gamma,\nu)\cdot n(x_\Gamma) := \mathrm{C}(x_\Gamma)\cdot n(x_\Gamma). \end{cases} \tag{1.3.26}$$

As second way to define this last operator is the following one: The restriction of this last operator of the tangent space $T_{x_\Gamma}\Gamma$ is defined by:

$$\mathcal{C}(x_\Gamma,\nu)|_{T_{x_\Gamma}\Gamma} := P(x_\Gamma)\begin{pmatrix} \dfrac{1+\nu\kappa_2}{1+\nu\kappa_1} & 0 \\ 0 & \dfrac{1+\nu\kappa_1}{1+\nu\kappa_2} \end{pmatrix} P(x_\Gamma)^{-1}.$$

This opeator is defined for $n(x\Gamma)$ by:

$$\mathcal{C}(x_\Gamma,\nu)n(x_\Gamma) := (1+\nu\kappa_1)(1+\nu\kappa_2)n(x_\Gamma).$$

Moreover we introduce the coefficient $\rho_\delta := \rho^\delta \circ \mathcal{L}^{-1}$ and $\mu_\delta := \mu^\delta \circ \mathcal{L}^{-1}$ and then we can state the following result:

Proposition 1.3.5. *The property* (1.3.19) *holds.*

Proof. Let $u_\delta, v_\delta \in H^1(\Omega_\delta)$, $u^\delta := v_\delta \circ \mathcal{L}$ and $u^\delta := v_\delta \circ \mathcal{L}$. Thanks to Proposition 1.3.3 we get:

$$\nabla u^\delta \cdot \nabla v^\delta = \big(\mathrm{C}^{-1}\,\mathcal{C}\,\nabla_\Gamma\, u_\delta \cdot \nabla_\Gamma\, v_\delta + \partial_\nu u_\delta \cdot \partial_\nu u_\delta\big) \circ \mathcal{L}.$$

From the definition of $\nabla_\mathcal{L}$ and $\mathcal{C}$ this last quantity become:

$$\nabla u^\delta \cdot \nabla v^\delta = \big[\mathrm{C}^{-1}(\mathcal{C}\,\nabla_\mathcal{L}\, u_\delta \cdot \nabla_\mathcal{L}\, v_\delta)\big] \circ \mathcal{L}.$$

This lead to :

$$\int_{C_{\delta,\eta_0}} \rho^\delta \nabla u^\delta \cdot \nabla v^\delta - \mu^\delta k^2 u^\delta \cdot v = \int_{C_{\delta,\eta_0}} \big(\mathrm{C}^{-1}\,Q\big) \circ \mathcal{L}$$

where Q defined on Ω_δ by:

$$Q := (\mathcal{C}\,\nabla_\mathcal{L}\, u_\delta \cdot \nabla_\mathcal{L}\, v_\delta) - k^2 \mu_\delta\,\mathrm{C}\,u_\delta \cdot v_\delta.$$

Thanks to Proposition 1.3.4 we have:

$$\int_{C_{\delta,\eta_0}} \big(\mathrm{C}^{-1}\,Q\big) \circ \mathcal{L} = \int_{\Omega_\delta} Q,$$

and then we get:

$$\int_{C_{\delta,\eta_0}} \rho^\delta \nabla u^\delta \cdot \nabla v^\delta - \mu^\delta k^2 u^\delta \cdot v = \int_{\Omega_\delta} \rho_\delta\,(\mathcal{C}\,\nabla_\mathcal{L}\, u_\delta \cdot \nabla_\mathcal{L}\, v_\delta) - \mu_\delta\,\mathrm{C}\,u_\delta \cdot v_\delta.$$

Thus we can conclude.

The right-hand-side $f_{\Sigma_{\eta_0}}$

As a direct consequence of (1.3.3) the linear form $f_{\Sigma_{\eta_0}}$ defined by:

$$\tilde{v}_\delta \mapsto \langle \partial_\nu u_f - \mathrm{DtN}\, u_f, \tilde{v}_\delta \circ \mathcal{L}\rangle_{\Gamma\times\{\eta_0\}},$$

is continuous on the space $H^{\frac{1}{2}}(\Gamma \times \{\eta_0\})$. Thus combining Proposition 1.3.5, with Proposition 1.3.2 yields the following result:

Lemma 1.3.6. *The function $u_\delta := u^\delta \circ \mathcal{L}^{-1}$ is the unique solution of the problem: Find $u_\delta \in H^1(\Omega_\delta)$ such that for all $v_\delta \in H^1(\Omega_\delta)$:*

$$a_\delta(u_\delta, v_\delta) = \langle f_{\Sigma_{\eta_0}}, v_\delta(\cdot,\eta_0)\rangle_{\Gamma\times\{\eta_0\}}. \tag{1.3.27}$$

1.3.2.2 Interpretation as a partial differential equation

Here we will transform the formulation (1.3.27) into a volume formulation. We introduce the operator $\operatorname{div}_{\mathcal{L}}$ defined for $\mathbf{u} : \Gamma \mapsto \mathbb{R}^3$ by:

$$\operatorname{div}_{\mathcal{L}}\mathbf{u} := \operatorname{div}_{\Gamma} \mathbf{u}_{\Gamma} + \partial_{\nu} u_{\nu},$$

where $\operatorname{div}_{\Gamma}$ is the surface divergence operator, $u_{\nu} := \mathbf{u} \cdot n$ and for all x_{Γ}, $\mathbf{u}_{\Gamma}(x_{\Gamma})$ is the projection of $\mathbf{u}$ on the tangent space $\mathrm{T}_{x_{\Gamma}}\Gamma$. According to [60, equation 2.5.205] the divergence operator $\operatorname{div}_{\Gamma}$ is defined for all $x_{\Gamma} \in \Gamma$ by:

$$\operatorname{div}_{\Gamma} u(x_{\Gamma}) := \frac{1}{\sqrt{\det(\mathrm{D}\,\phi_{x_{\Gamma}}^{\dagger}(0) \cdot \mathrm{D}\,\phi_{x_{\Gamma}}(0))}} \operatorname{div}\left(\sqrt{\det(\mathrm{D}\,\phi_{x_{\Gamma}}^{\dagger} \cdot \mathrm{D}\,\phi_{x_{\Gamma}})}\, \mathrm{D}\,\phi_{x_{\Gamma}}^{-1} \cdot u_{\Gamma} \circ \phi_{x_{\Gamma}}^{-1}\right)(0),$$

and thanks to [60, Theorem 2.5.19] this last operator satisfies the green formula:

$$\forall \boldsymbol{u}, v, \int_{\Gamma} \boldsymbol{u} \cdot \nabla_{\Gamma} v d\Gamma = -\int_{\Gamma} \operatorname{div}_{\Gamma} u \cdot \overline{v} d\Gamma,$$

From this last formula we get that u_{δ} satisfies:

$$\operatorname{div}_{\mathcal{L}}\big(\rho_{\delta}\, \mathcal{C}\, \nabla_{\mathcal{L}}\, u_{\delta}\big) + k^2 \mu_{\delta}\, \mathrm{C}\, u_{\delta} = 0 \quad \text{in } \Omega_{\delta}.$$

Moreover since for all $x_{\Gamma} \in \Gamma$, $n(x_{\Gamma})$ is an eigenvector of $\mathcal{C}$, u_{δ} satisfies the boundary condition:

$$\partial_{\nu} u_{\delta} = 0 \quad \text{on} \quad \Gamma \times \{-\delta\}.$$

Finally our solution u_{δ} satisfies the following boundary condition:

$$\partial_{\nu} u_{\delta} - \mathrm{DtN}_{\mathcal{L}}\, u_{\delta} = f_{\Sigma_{\eta_0}} \quad \text{on} \quad \Gamma \times \{\eta_0\}.$$

Chapter 2

Formal asymptotic expansion

We wish to construct here a (formal) approximation of the solution u^δ of (1.1.1) of the form $u^0 + \delta u^1 + \delta^2 u^2 + \cdots$ where for all i, $u^i : \Omega \mapsto \mathbb{C}$. Numerical simulation show that the true solution u^δ strongly oscillates in the neighborhood of the boundary Γ(See Figure 2.1). Such a phenomenon is what we call a boundary layer phenomenon. Hence $u^0 + \delta u^1 + \delta^2 u^2 + \cdots$ will be only valid far from the interface Γ. (See Figure 2.1) To take into account this boundary layer phenomenon, we chooze to apply the matched asymptotic expansion method. For an example this method is explained, for instance, in [33, 70, 46, 54, 41, 1, 68] and applied in [48, 49, 31, 16, 47, 15, 37, 34, 35, 38].

First we will assume that the expansion:

$$u^\delta = u^0 + \delta u^1 + \delta^2 u^2 + \cdots , \tag{2.0.1}$$

is valid far from the interface Γ. We will formally see that for all i, u^i is a solution of the Helmholtz equation and then it remains to determine the trace of u_i on the interface Γ in order to completely determine the term u_i.

Secondly, to represent the strong oscillations of the function u^δ in the neighbourhood of Γ, we will assume for x near the interface that:

$$u^\delta(x) = \hat{u}_0(x_\Gamma; \hat{x}, \hat{\nu}) + \delta \hat{u}_1(x_\Gamma; \hat{x}, \hat{\nu}) + \delta^2 \hat{u}_2(x_\Gamma; \hat{x}, \hat{\nu}) + \cdots , \tag{2.0.2}$$

where $(\hat{x}, \hat{\nu}) := \big(\psi_\Gamma(x_\Gamma), \nu\big)/\delta$ and x_Γ is the nearest point in Γ from x and ν is the distance of x from Γ. According to Proposition 1.3.1, the quantity (x_Γ, ν) is also the unique solution of $x = x_\Gamma + \nu n(x_\Gamma)$ and the local coordinates is a diffeomorphism.

The function $\hat{u}_i(x_\Gamma; \hat{x}, \hat{\nu})$ are searched as periodic of period $T = 1$ in $\hat{x}$ for all i : This is the ansatz of homogenization theory (see [4, 3, 5, 6, 12, 18, 29, 66]). Inserting this expansion into the Helmholtz equation yields for all $n \in \mathbb{N}$ a recursive equation between $\hat{u}_n(x_\Gamma; \hat{x}, \hat{\nu})$ and the previous terms $\hat{u}_{n-1}(x_\Gamma; \hat{x}, \hat{\nu}), \cdots$. Nevertheless this equation will not completely determine the near field $(x_\Gamma; \hat{x}, \hat{\nu}) \mapsto \hat{u}_n(x_\Gamma; \hat{x}, \hat{\nu})$: the operator $\mathcal{T}_0$ to be inverted at each step of rhe recurrence has a on trivial kernel namely the space of function $(x_\Gamma, \hat{x}, \hat{\nu}) \mapsto v(x_\Gamma, \hat{x}, \hat{\nu})$ which are independent of $(\hat{x}, \hat{\nu})$ i.e.

$$\exists V : x_\Gamma \mapsto V(x_\Gamma),\ \forall x_\Gamma,\ \forall (\hat{x}, \hat{\nu}),\ v(x_\Gamma; \hat{x}, \hat{\nu}) = V(x_\Gamma).$$

The two expansions (2.0.1) and (2.0.1) have to be both valid in what we call the matching zone which brings us equation between the Taylor expansion $u^n, u^{n-1}, \cdots$ and

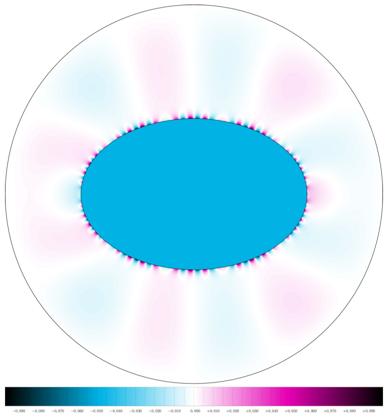


Figure 2.1: The boundary layer phenomenon

the asymptotic behavior of $\hat{u}_n(x_\Gamma; \hat{x}, \hat{\nu})$ for large $\hat{\nu}$. In particular these equations will fix the function of x_Γ in the kernel of T for the term $\hat{u}_n$. These equations are called the "matching conditions".

Finally we will construct an explicit recursive algorithm that construct iteralively the two sequence $(u_n)_n$ and $(\hat{u}_n)_n$.

2.1 Equations of the problem

We recall that the goal of this work is to construct approximations of the unique solution $u_\delta : \Omega_\delta \mapsto \mathbb{C}$ of:

$$\mathrm{div}_{\mathcal{L}}\big(\rho_\delta\, \mathcal{C}\, \nabla_{\mathcal{L}}\, u_\delta\big) + k^2 \mu_\delta\, \mathrm{C}\, u_\delta = 0 \quad \text{in } \Omega_\delta, \tag{2.1.3}$$

with the following boundary condition:

$$\partial_\nu u_\delta = 0 \quad \text{on} \quad \Gamma \times \{0\} \tag{2.1.4}$$

and

$$\partial_\nu u_\delta - \mathrm{DtN}_{\mathcal{L}}\, u_\delta = f_{\Sigma_{\eta_0}} \quad \text{on} \quad \Gamma \times \{\eta_0\}. \tag{2.1.5}$$

We recall that the operator $\mathrm{DtN}_{\mathcal{L}}$ is defined by (1.3.25).

2.2 Quick presentation of the matched asymptotic expansion method

This method consists in seeking two asymptotic expansions of the solution. One(called far-field expansion) is valid near the boundary called the near-field expansion and the other one is valid far from the boundary . Firstly let us chose a function $\eta : \delta \mapsto \eta(\delta)$ such that:

$$\lim_{\delta \to \infty} \eta(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to \infty} \frac{\eta(\delta)}{\delta} = \infty, \tag{2.2.6}$$

and define the following zones(see Figure 2.2 for a graphical illustration of these regions):

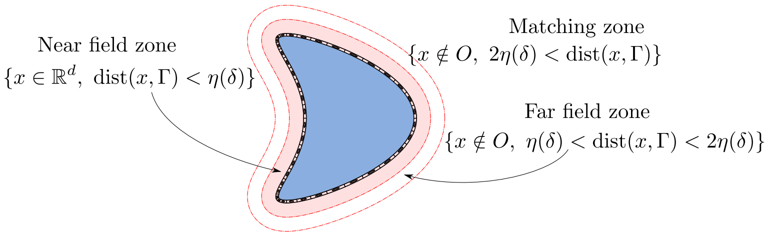


Figure 2.2: The three zones

- The near-field zone is defined by: $\Gamma\times]-\delta,\eta[$. In this zone we formally assume that u_δ is a series of $\psi_\Gamma -\delta-$periodic functions(See Theorem 1.2.5 for the defintion of the $\psi_\Gamma -\delta-$periodicity) in the sense that for all $N \in \mathbb{N}$ there exists a constant C_N such that the following estimate holds:
$$\|u_\delta - u_\delta^N\| \leq C_N \delta^{N+1}, \tag{2.2.7}$$
where u_δ^N is defined for $(x_\Gamma,\nu) \in \Gamma\times]-\delta,\eta[$ by:
$$u_\delta^N(x_\Gamma,\nu) = \sum_{n=0}^{N} \delta^n \hat{u}_n(x_\Gamma;\hat{x},\hat{\nu}) \quad \text{with} \quad (\hat{x},\hat{\nu}) := \frac{(\psi_\Gamma(x_\Gamma),\nu)}{\delta}.$$
In this last defintion, for all n in $\mathbb{N}$ the application $\hat{u}_n : \Gamma \mapsto P(\hat{\Omega})$ is defined from Γ into $P(\hat{\Omega})$ and we recall that $P(\hat{\Omega})$ is the set of functions defined on $\hat{\Omega} := \mathbb{R}^2\times]-1,\infty[$ and one periodic on the variable $\hat{x}$.

- The far-field zone is defined by: $\Gamma\times]2\eta,\eta_0[$. In this zone we assume that u_δ admits the following asymptotic expansion: For all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that the following estimate holds:
$$\|u_\delta - u_\delta^N\| \leq C_N \delta^{N+1}, \tag{2.2.8}$$
where u_δ^N is defined for $(x_\Gamma,\nu) \in \Gamma\times]2\eta,\eta_0[$ by:
$$u_\delta^N(x_\Gamma,\nu) = \sum_{n=0}^{N} \delta^n u_n(x_\Gamma,\nu),$$
and for all $n \in \mathbb{N}$ the function u_n is defined on Ω_0.

- The overlapping zone is defined by: $\Gamma\times]\eta,2\eta[$. In this zone expansions (2.2.7) and (2.2.8) are assumed to be both valid and then should be equivalent.

2.3 Identification of the required equations for the ansatz

2.3.1 Equations of the near-field

We require that the near field satisfies (2.1.3) and the Neumann condition (2.1.4). Following [37] and [34] we chose that:
$$\boxed{\forall n \in \mathbb{N},\ \partial_{\hat{\nu}} \hat{u}_n = 0 \text{ on } \Gamma \times \partial\hat{\Omega}}. \tag{2.3.9}$$

In the homogenization theory in [5, 4, 3], in order to perform a formal computation, the authors strongly use the following property : For all u^δ, A^δ of the form:

$$u^\delta(x) = u\left(x, \frac{x}{\delta}\right) \quad \text{and} \quad A^\delta(x) = A\left(x, \frac{x}{\delta}\right)$$

we have for all x that:

$$\left\{\begin{aligned} \operatorname{div}\left(A^\delta \nabla u^\delta\right)(x) = \delta^{-2}\operatorname{div}_y\big(A(x,y)\nabla_y u\big)(x,y) \\ + \delta^{-1}\operatorname{div}_y\big(A(x,y)\nabla_x u\big)(x,y) + \delta^{-1}\operatorname{div}_x\big(A(x,y)\nabla_y u\big)(x,y) \\ + \operatorname{div}_x\big(A(x,y)\nabla_x u\big)(x,y) \end{aligned}\right. \tag{2.3.10}$$

where $y := x/\delta$. They formally deduce from these equalities that if a function u^δ takes the form

$$u^\delta(x) = \sum_{n\in\mathbb{N}} \delta^n u_n\left(x, \frac{x}{\delta}\right),$$

and satisfies $\operatorname{div}(A^\delta u^\delta) = f$ then the sequence u_n satisfies the following induction equality:

$$\left\{\begin{aligned} 0 =& \operatorname{div}_y\big(A(x,y)\nabla_y u_n\big)(x,y) + \\ & \operatorname{div}_x\big(A(x,y)\nabla_y u_{n-1}\big)(x,y) + \operatorname{div}_y\big(A(x,y)\nabla_x u_{n-1}\big)(x,y) + \\ & \operatorname{div}_x\big(A(x,y)\nabla_x u_{n-2}\big)(x,y). \end{aligned}\right. \tag{2.3.11}$$

This last formula completely defines all term u_n, and then builds the expansion $u^\delta \approx u_0 + \delta u_1 + \cdots$.

Since the $\psi_\Gamma - \delta-$periodicity is a generalization of the periodicity, we will draw on this last idea for the construction of the near field. More precisely, we now extend the expression (2.3.10) for $\psi_\Gamma - \delta-$periodic function and the operator:

$$\operatorname{div}_{\mathcal{L}}\big(\rho_\delta\, \mathcal{C}\, \nabla_{\mathcal{L}} \cdot\big) + k^2 \mu_\delta\, \mathrm{C} \cdot,$$

and that will be the object of the formula (2.3.23). In order to simplify notation we introduce the operator $\mathcal{I}_\delta$ defined for $\hat{u} : \Gamma \mapsto P(\hat{\Omega})$ and $(x_\Gamma, \nu) \in \Omega_\delta$ by:

$$\mathcal{I}_\delta\, \hat{u}(x_\Gamma, \nu) := \hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) \quad \text{with} \quad (\hat{x}, \hat{\nu}) := \frac{(\psi_\Gamma(x_\Gamma), \nu)}{\delta}.$$

The formula (2.3.10) is a combination of the two following formula:

$$\forall x,\ u^\delta(x) = u\left(x, \frac{x}{\delta}\right) \Longrightarrow \forall x,\ \nabla u^\delta(x) = \nabla_x u(x,y) + \delta^{-1}\nabla_y u(x,y), \tag{2.3.12}$$

and

$$\forall x,\ u^\delta(x) = u\left(x, \frac{x}{\delta}\right) \Longrightarrow \forall x,\ \operatorname{div} u^\delta(x) = \operatorname{div}_x u(x,y) + \delta^{-1}\operatorname{div}_y u(x,y), \tag{2.3.13}$$

where $y := x/\delta$.

Thus we will extend these two formulas for function $\psi_\Gamma - \delta-$periodic. That is the object of Proposition 2.3.1 which is an extension of (2.3.12) and Proposition 2.3.2 which

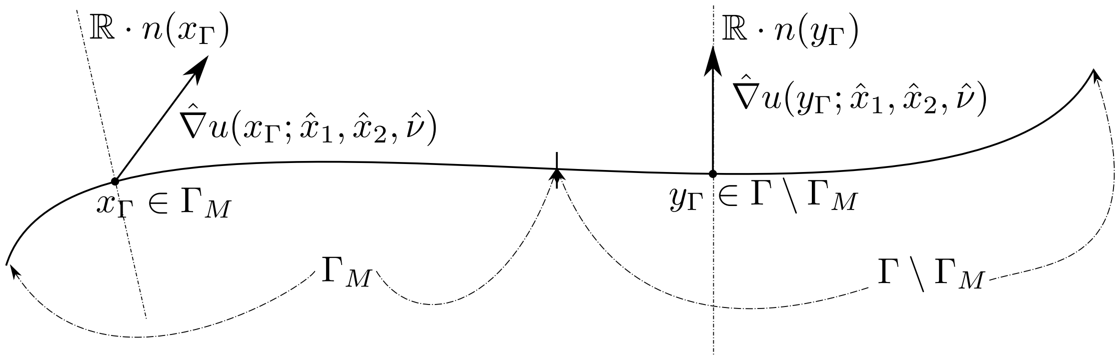


Figure 2.3: Illustration of the operator $\widehat{\nabla}$

is an extension of (2.3.13). To state these two results, one needs to introduce the operator $\widehat{\nabla}$ defined for $\hat{u} \in \Gamma \times \hat{\Omega} \to \mathbb{R}$ and $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma_\mathrm{M} \times \hat{\Omega}$ by :

$$\widehat{\nabla}\,\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) := \mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger \nabla_{\hat{x}}\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) + \partial_{\hat{\nu}}\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\cdot n(x_\Gamma). \tag{2.3.14}$$

Here $\nabla_{\hat{x}}$ is the classical gradient with respect to $\hat{x}$ defined by:

$$\nabla_{\hat{x}}\hat{u} := \begin{pmatrix} \partial_{\hat{x}_1}\hat{u} \\ \partial_{\hat{x}_2}\hat{u} \end{pmatrix}.$$

Nevertheless, the function ψ_Γ is a priori non regular for $x_\Gamma \notin \Gamma_\mathrm{M}$ and then the quantity $\mathrm{D}\,\psi_\Gamma(x_\Gamma)$ is not defined. Hence (2.3.14) does not have sense . However, according to the "patching solution" our coefficients vary slowly in $(\Gamma\setminus\Gamma_\mathrm{M})\times]-\delta,0[$ and then we can prove that the same applies for the solution u_δ. Therefore for all $x_\Gamma \in \Gamma\setminus\Gamma_\mathrm{M}$ and $(\hat{x},\hat{\nu}) \in \hat{\Omega}$ all quantity $\hat{u}(x_\Gamma;\hat{x},\hat{\nu})$ will not depend of the variable $\hat{x}$. Then we extend $\widehat{\nabla}\,\hat{u}$ for $\big(x_\Gamma,\hat{x},\hat{\nu}\big) \in \Gamma\setminus\Gamma_\mathrm{M}\times\hat{\Omega}$ by $\widehat{\nabla}\,\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) = \partial_{\hat{\nu}}\hat{u}(x_\Gamma;\hat{x},\hat{\nu})$. (See Figure 2.3) Next we define the operator $\widehat{\mathrm{div}}$ for a vector function $\hat{\boldsymbol{u}} : \Gamma\times\hat{\Omega} \mapsto \mathbb{R}^3$ for $(x_\Gamma,\hat{x},\hat{\nu}) \in \Gamma_\mathrm{M}\times\hat{\Omega}$ by:

$$\widehat{\mathrm{div}}\,\big(\hat{\boldsymbol{u}}(x_\Gamma;\hat{x},\hat{\nu})\big) := \mathrm{div}_{\hat{x}}\left(\mathrm{D}\,\psi_\Gamma(x_\Gamma,\hat{x},\hat{\nu})\hat{\boldsymbol{u}}_\Gamma(x_\Gamma,\hat{x},\hat{\nu})\right) + \partial_{\hat{\nu}}\hat{u}_\nu(x_\Gamma;\hat{x},\hat{\nu}). \tag{2.3.15}$$

Here we defined $u_\nu(x_\Gamma;\hat{x},\hat{\nu}) := \hat{\mathbf{u}}(x_\Gamma;\hat{x},\hat{\nu})\cdot n(x_\Gamma)$ and $\hat{\mathbf{u}}_\Gamma(x_\Gamma;\hat{x},\hat{\nu})$ is the projection of $\hat{\mathbf{u}}$ on the tangent space $\mathrm{T}_{x_\Gamma}\Gamma$ and $\mathrm{div}_{\hat{x}}$ is the classical divergence with respect to $\hat{x}$ defined for function $\boldsymbol{u} := (u_1,u_2)$ by $\mathrm{div}_{\hat{x}}\boldsymbol{u} := \partial_{\hat{x}_1}u_1 + \partial_{\hat{x}_2}u_2$.

Nevertheless, since the function ψ_Γ is a propri non regular for $x_\Gamma \in \Gamma\setminus\Gamma_\mathrm{M}$ (2.3.15) then (2.3.15) does not have sense. However we have seen that for $(\hat{x},\hat{\nu}) \in \hat{\Omega}$ all quantity $\hat{\boldsymbol{u}}(x_\Gamma;\hat{x},\hat{\nu})$ will not depend of the variable $\hat{x}$. Then we extend $\widehat{\mathrm{div}}\,\hat{\boldsymbol{u}}(x_\Gamma;\hat{x},\hat{\nu})$ for $(x_\Gamma,\hat{x},\hat{\nu}) \in (\Gamma\setminus\Gamma_\mathrm{M})\times\hat{\Omega}$ by $\widehat{\mathrm{div}}\,\hat{\boldsymbol{u}}(x_\Gamma;\hat{x},\hat{\nu}) := \partial_{\hat{\nu}}\hat{u}_\nu(x_\Gamma;\hat{x},\hat{\nu})$.

Now let us give the expressions of these two operators in the case of the unit sphere when the map ψ_Γ is the spherical coordinate. We recall that it mean that the map ψ_Γ is defined for $x_\Gamma \in \Gamma$ by:

$$\psi_\Gamma(x_\Gamma) := (\theta,\phi),$$

where (θ,ϕ) is the unique solution of of $x_\Gamma = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$.(See Figure 1.13).

In this case, thanks to (1.2.12), we can prove that the map $\mathrm{D}\,\psi_\Gamma{}^\dagger$ is given for $x_\Gamma \in \Gamma$ by:

$$\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger(x_\Gamma) = \begin{pmatrix} -\dfrac{\cos(\phi)\sin(\theta)}{\cos(\theta)^2} & -\sin(\phi)\cos(\theta) \\ \dfrac{\cos(\phi)}{\cos(\theta)} & -\sin(\phi)\sin(\theta) \\ 0 & \cos(\phi) \end{pmatrix} \quad \text{and} \quad n(x_\Gamma) = \begin{pmatrix} \cos(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) \\ \sin(\phi) \end{pmatrix}.$$

Therefore in this case, thanks to (1.2.15), if $-\eta < \phi < \eta$ then we have for $u : \Gamma \times \hat{Y}_\infty$ smooth enough:

$$(\widehat{\nabla}\, u)(x_\Gamma;\cdot) = v_1 \partial_{\hat{x}_1} u(x_\Gamma;\cdot) + v_2 \partial_{\hat{x}_2} u(x_\Gamma;\cdot) + v_{\hat{\nu}} \partial_{\hat{\nu}} u(x_\Gamma;\cdot),$$

where:

$$v_1 := \begin{pmatrix} -\dfrac{\cos(\phi)\sin(\theta)}{\cos(\theta)^2} \\ \dfrac{\cos(\phi)}{\cos(\theta)} \\ 0 \end{pmatrix}, \; v_2 := \begin{pmatrix} -\sin(\phi)\cos(\theta) \\ -\sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix} \quad \text{and} \quad v_{\hat{\nu}} := \begin{pmatrix} \cos(\phi)\cos(\theta) \\ \cos(\phi)\sin(\theta) \\ \sin(\phi) \end{pmatrix}$$

else one has:

$$(\widehat{\nabla}\, u)(x_\Gamma;\cdot) = v_{\hat{\nu}} \partial_{\hat{\nu}} u.$$

For vector field $\boldsymbol{u} : \Gamma \times \hat{\Omega}$ smooth enough, if $-\eta < \phi < \eta$ then

$$\widehat{\operatorname{div}}(\boldsymbol{u})(x_\Gamma;\cdot) = \partial_{\hat{x}_1}(v_1, u(x_\Gamma;\cdot)) + \partial_{\hat{x}_2}(v_2, u(x_\Gamma;\cdot)) + \partial_{\hat{\nu}}(v_{\hat{\nu}}, u(x_\Gamma;\cdot)),$$

else $\widehat{\operatorname{div}}(\boldsymbol{u})(x_\Gamma;\cdot) = \partial_{\hat{\nu}}(v_{\hat{\nu}}, u(x_\Gamma;\cdot))$.

Thus thanks to these definitions we can state the following result:

Proposition 2.3.1. *Let $\hat{u} \in C^1(\Gamma \times \hat{\Omega})$ be a function patching-ψ_Γ-admissible then the following equality holds:*

$$\nabla_{\mathcal{L}}(\mathcal{I}_\delta\, \hat{u}) = \mathcal{I}_\delta \Big(\nabla_\Gamma\, \hat{u} + \delta^{-1}\, \widehat{\nabla}\, \hat{u} \Big).$$

Proposition 2.3.2. *Let $\hat{\mathbf{u}} \in \Big(C^1(\Gamma \times \hat{\Omega})\Big)^3$ patching-ψ_Γ-admissible then the following equality holds:*

$$\operatorname{div}_{\mathcal{L}}\big(\mathcal{I}_\delta\, \hat{\mathbf{u}}\big) = \mathcal{I}_\delta \Big(\operatorname{div}_\Gamma \hat{\mathbf{u}}_\Gamma + \delta^{-1}\, \widehat{\operatorname{div}}\, \hat{\mathbf{u}} \Big),$$

where for all x_Γ, $\mathbf{u}_\Gamma(x_\Gamma;\cdot)$ is the projection of $\mathbf{u}(x_\Gamma;\cdot)$ on the tangent space $\mathrm{T}_{x_\Gamma}\Gamma$.

Proof of Proposition 2.3.1. First let us prove that for all $(x_\Gamma, \nu) \in \Omega_\delta$ we have:

$$\nabla_\Gamma \big(\mathcal{I}_\delta(\hat{u})\big)(x_\Gamma, \nu) = \nabla_\Gamma(\hat{u})(x_\Gamma; \hat{x}, \hat{\nu}) + \delta^{-1}\, \mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger \hat{\nabla} \hat{u}(x_\Gamma; \hat{x}, \hat{\nu}), \tag{2.3.16}$$

where: $(\hat{x}, \hat{\nu}) := \dfrac{(\psi_\Gamma(x_\Gamma), \nu)}{\delta}$.

Indeed, assume first that $x_\Gamma \in \Gamma_{\mathrm{M}}$. Then from the definition of Γ_{M} the application ψ_Γ is differentiable and bijective from a neighborhood $V(x_\Gamma) \subset \Gamma$ of x_Γ into a neighborhood

$V(x_r) \subset \mathbb{R}^2$ of $x_r := \psi_\Gamma(x_\Gamma)$. Thus its inverse $\phi_\Gamma := {\psi_\Gamma}^{-1} : V(x_r) \mapsto V(x_\Gamma)$ is a local parameterization of the surface Γ and the family $(e_i)_{i\in\{1,2\}}$ defined for $i = 1, 2$ by:

$$e_i := \mathrm{D}\,\phi_\Gamma(x_r)\hat{e}_i, \tag{2.3.17}$$

is a basis of $T_{x_\Gamma}\Gamma$. Here $(\hat{e}_i)_{i=1,2}$ is the canonical basis of $\mathbb{R}^2$.

For all $\nu \in]-\delta, \eta_0[$ the following decomposition of the map $x_\Gamma \mapsto \mathcal{I}_\delta(\hat{u})(x_\Gamma, \nu)$ holds:

$$\mathcal{I}_\delta(\hat{u})(\cdot, \nu) = u^\delta_{\psi_\Gamma,\nu} \circ \psi_\Gamma,$$

where $u^\delta_{\psi_\Gamma,\nu}$ is defined for $x \in V(x_r)$ by:

$$u^\delta_{\psi_\Gamma,\nu}(x) := \hat{u}\left(\phi_\Gamma(x), \frac{x}{\delta}, \hat{\nu}\right), \tag{2.3.18}$$

which yields for all $i \in \{1, 2\}$:

$$\nabla_\Gamma(\mathcal{I}_\delta\,\hat{u})(x_\Gamma, \nu) \cdot e_i = \partial_{x_i} u^\delta_{\psi_\Gamma,\nu}(x_r). \tag{2.3.19}$$

We define the function $\hat{u}_{\psi_\Gamma,\nu}$ for $(x, \hat{x}) \in V(x_r) \times \hat{\Omega}$ by:

$$\hat{u}_{\psi_\Gamma,\nu}(x, \hat{x}) := \hat{u}(\phi_\Gamma(x), \hat{x}). \tag{2.3.20}$$

Thanks to this definition, we can rewrite (2.3.18) for all $x \in V(x_r)$ as follow:

$$u^\delta_{\psi_\Gamma,\nu}(x) = \hat{u}_{\psi_\Gamma,\nu}\left(x, \frac{x}{\delta}\right),$$

which leads to:

$$\partial_{x_i} u^\delta_{\psi_\Gamma,\nu}(x_r) = \partial_{x_i}\hat{u}_{\psi_\Gamma,\nu}(x_r, \hat{x}) + \delta^{-1}\partial_{\hat{x}_i}\hat{u}_{\psi_\Gamma,\nu}(x_r, \hat{x}) \tag{2.3.21}$$

From (2.3.20), we get the following decomposition of the map $x_\Gamma \mapsto \hat{u}(x_\Gamma; \hat{x}, \hat{\nu})$:

$$\hat{u}(\cdot; \hat{x}, \hat{\nu}) = \hat{u}_{\psi_\Gamma,\nu}(\cdot, \hat{x}) \circ \psi_\Gamma,$$

which leads to:

$$\nabla_\Gamma\,\hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) \cdot e_i = \partial_{x_i}\hat{u}_{\psi_\Gamma,\nu}(x_r, \hat{x}). \tag{2.3.22}$$

Thanks to (2.3.20) we get $\partial_{\hat{x}_i}\hat{u}_{\psi_\Gamma,\nu}(x_r, \hat{x}) = \partial_{\hat{x}_i}\hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) = \big(\hat{\nabla}_{\hat{x}}\hat{u}(x_\Gamma; \hat{x}, \hat{\nu})\big) \cdot \hat{e}_i$ and using (2.3.17) yields:

$$\partial_{\hat{x}_i}\hat{u}_{\psi_\Gamma,\nu}(x_r, \hat{x}) = \hat{\nabla}_{\hat{x}}\hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) \cdot \big(\mathrm{D}\,\psi_\Gamma(x_\Gamma)e_i\big) = \big(\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger\hat{\nabla}_{\hat{x}}\hat{u}\big)(x_\Gamma; \hat{x}, \hat{\nu}) \cdot e_i$$

Combining this with (2.3.22), (2.3.21) and (2.3.19) conclude the proof of (2.3.16) when $x_\Gamma \in \Gamma_\mathrm{M}$. For $x_\Gamma \notin \Gamma_\mathrm{M}$ this result is trivial because $u(x_\Gamma; \cdot)$ only depend of $\hat{\nu}$.

Finally, we have, $\forall (x_\Gamma, \nu) \in \Omega_\delta$:

$$\partial_\nu\left(\hat{u}\Big(x_\Gamma; \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\Big)\right) = \delta^{-1}\partial_{\hat{\nu}}\hat{u}(x_\Gamma; \hat{x}, \hat{\nu}),$$

and multiplying this last identity by $n(x_\Gamma)$ and combining with (2.3.16) conclude the proof. $\square$

Proof of Proposition 2.3.2. The proof use similar calculations. Indeed it is sufficient to replace (2.3.19) by

$$\operatorname{div}_\Gamma\Big(\mathcal{I}_\delta\,\mathbf{u}_\Gamma\Big)(x_\Gamma) = \sum_{j=1}^{2}\frac{1}{\sqrt{g}}\partial_{x_i}\Big(\sqrt{g}\big(\boldsymbol{u}_\delta^{\psi_\Gamma}\big)^i\Big)(x_r).$$

Here, we defined on $V(x_r)$ the scalar functions $\sqrt{g}$ as the determinant of the matrix whose coefficient are given for all $i,j=1,2$ by $\partial_{x_i}\phi_\Gamma\cdot\partial_{x_j}\phi_\Gamma$. and $\big(\boldsymbol{u}_\delta^{\psi_\Gamma}\big)^i$ are the unique scalars such that:

$$\boldsymbol{u}_\delta^{\psi_\Gamma} := \boldsymbol{u}_\delta\circ\phi_\Gamma = \sum_{i=1}^{2}\big(\boldsymbol{u}_\delta^{\psi_\Gamma}\big)^i\partial_{x_i}\phi_\Gamma$$

Our goal is to give a formula which is the equivalent of formula (2.3.10) in the case of the plane and $k=0$; i.e. to give formula of the form for all $\hat{u}:\Gamma\mapsto P(\hat{\Omega})$ smooth enough:

$$\operatorname{div}_{\mathcal{L}}\big(\rho_\delta\,\mathcal{C}\,\nabla_{\mathcal{L}}(\mathcal{I}_\delta\,\hat{u})\big) + k^2\mu_\delta\,\mathrm{C}\,\mathcal{I}_\delta\,\hat{u} = \mathcal{I}_\delta\,\big(\mathcal{T}_\delta\hat{u}\big), \tag{2.3.23}$$

where $\mathcal{T}_\delta$ is an operator to be determined at least formally. If $\mathcal{C}$ and C were a constant, using:

$$\nabla_{\mathcal{L}}(\mathcal{I}_\delta\,\hat{u}) = \mathcal{I}_\delta\Big(\nabla_\Gamma\,\hat{u} + \delta^{-1}\,\widehat{\nabla}\,\hat{u}\Big) \quad\text{and}\quad \operatorname{div}_{\mathcal{L}}\big(\mathcal{I}_\delta\,\hat{\mathbf{u}}\big) = \mathcal{I}_\delta\Big(\operatorname{div}_\Gamma\,\hat{\mathbf{u}}_\Gamma + \delta^{-1}\,\widehat{\operatorname{div}}\,\hat{\mathbf{u}}\Big).$$

(See Proposition 2.3.1 and Proposition 2.3.2), we would get, since $\mathcal{C}\,\mathcal{I}_\delta = \mathcal{I}_\delta\,\mathcal{C}$ and $\mathrm{C}\,\mathcal{I}_\delta = \mathcal{I}_\delta\,\mathrm{C}$ ($\mathcal{C}$ and C both commute with $\mathcal{I}_\delta$):

$$\mathcal{T}_\delta = \big(\delta^{-1}\,\widehat{\operatorname{div}} + \operatorname{div}_\Gamma\big)\Big(\mathcal{C}\,\big(\delta^{-1}\,\widehat{\nabla} + \nabla_\Gamma\big)\Big) + k^2\hat{\mu}\,\mathrm{C}\,. \tag{2.3.24}$$

In the general case, one has:

$$\begin{aligned}
\mathcal{C}(x_\Gamma,\nu)\,\mathcal{I}_\delta\,\hat{\boldsymbol{u}}(x_\Gamma,\nu) &= \mathcal{C}(x_\Gamma,\nu)\hat{\boldsymbol{u}}\left(x_\Gamma;\frac{\psi_\Gamma(x_\Gamma)}{\delta},\frac{\nu}{\delta}\right),\\
&= \mathcal{C}^\delta\left(x_\Gamma,\frac{\nu}{\delta}\right)\hat{\boldsymbol{u}}\left(x_\Gamma;\frac{\psi_\Gamma(x_\Gamma)}{\delta},\frac{\nu}{\delta}\right),\\
&= \mathcal{I}_\delta(C^\delta\hat{\boldsymbol{u}})(x_\Gamma,\nu),
\end{aligned}$$

where $\mathcal{C}^\delta(x_\Gamma;\hat{x},\hat{\nu}) := \mathcal{C}(x_\Gamma;\delta\hat{\nu})$. Posing $\mathrm{C}^\delta(x_\Gamma;\hat{x},\hat{\nu}) := \mathrm{C}(x_\Gamma;\delta\hat{\nu})$, we also have:

$$\mathrm{C}(x_\Gamma,\nu)\,\mathcal{I}_\delta\,\hat{u}(x_\Gamma,\nu) = \mathcal{I}_\delta(\mathrm{C}^\delta\,\hat{\boldsymbol{u}})(x_\Gamma,\nu).$$

Hence (2.3.24) becomes in the general case:

$$\mathcal{T}_\delta = \big(\delta^{-1}\,\widehat{\operatorname{div}} + \operatorname{div}_\Gamma\big)\Big(\mathcal{C}^\delta\,\big(\delta^{-1}\,\widehat{\nabla} + \nabla_\Gamma\big)\Big) + k^2\hat{\mu}\,\mathrm{C}^\delta\,. \tag{2.3.25}$$

Our next goal will be to express an expansion of $\mathcal{T}_\delta$ in power of δ in the form:

$$\mathcal{T}_\delta = \sum_{k\in\mathbb{Z}}\delta^{k-2}\mathcal{T}_k, \tag{2.3.26}$$

and we want to identify the $\mathcal{T}_k$. For this, using the Taylor expansion formula, we write:

$$\mathcal{C}(x_\Gamma;\nu) = \sum_{k\in\mathbb{N}} \nu^k \mathcal{C}^{(k)}(x_\Gamma) \quad \text{and} \quad \mathrm{C}(x_\Gamma;\nu) = \sum_{k\in\mathbb{N}} \nu^k c^{(k)}(x_\Gamma).$$

Here we defined for $k \in \mathbb{Z}$ and $x_\Gamma \in \Gamma$, the following quantity:

$$\mathcal{C}^{(k)}(x_\Gamma) := \frac{1}{k!}\partial_\nu^k \,\mathcal{C}(x_\Gamma, 0) \quad \text{if} \quad k \geq 0 \quad \text{else} \quad \mathcal{C}^{(k)}(x_\Gamma) := 0, \tag{2.3.27}$$

and

$$c^{(k)}(x_\Gamma) := \frac{1}{k!}\partial_\nu^k \,\mathrm{C}(x_\Gamma, 0) \quad \text{if} \quad k \geq 0 \quad \text{else} \quad c^{(k)}(x_\Gamma) := 0. \tag{2.3.28}$$

Then according to the definition of $\mathcal{C}^\delta$ and C^δ, we have:

$$\mathcal{C}^\delta(x_\Gamma;\hat{\nu}) = \sum_{k\in\mathbb{N}} \delta^k \hat{\nu}^k \mathcal{C}^{(k)}(x_\Gamma) \quad \text{and} \quad \mathrm{C}^\delta(x_\Gamma;\hat{\nu}) = \sum_{k\in\mathbb{N}} \delta^k \hat{\nu}^k c^{(k)}(x_\Gamma).$$

Thanks to these to equalities, (2.3.25) formally becomes:

$$\mathcal{T}_\delta = \mathcal{T}_\delta^\rho + k^2 \hat{\mu} \sum_{k\in\mathbb{N}} \delta^k \hat{\nu}^k c^{(k)}(x_\Gamma),$$

where we defined for u smooth enough:

$$\mathcal{T}_\delta^\rho \hat{u} := \sum_{k\in\mathbb{N}} \left(\delta^{-1}\,\widehat{\mathrm{div}} + \mathrm{div}_\Gamma\right) \Big(\delta^k \hat{\nu}^k \mathcal{C}^{(k)}(x_\Gamma)\big(\delta^{-1}\,\widehat{\nabla} + \nabla_\Gamma\big)\Big).$$

Thus it remains to express an expansion of $\mathcal{T}_\delta^\rho$ in powers of δ of the form:

$$\mathcal{T}_\delta^\rho = \sum_{k\in\mathbb{Z}} \delta^{k-2} \mathcal{T}_k^\rho. \tag{2.3.29}$$

Using an index rearrangement, we have for $\hat{u}$:

$$\sum_{k\in\mathbb{Z}} \delta^k \hat{\nu}^k \mathcal{C}^{(k)} \big(\nabla_\Gamma \hat{u} + \delta^{-1}\,\widehat{\nabla}\,\hat{u}\big) = \sum_{k\in\mathbb{Z}} \delta^k \Big(\hat{\nu}^k \mathcal{C}^{(k)}\,\nabla_\Gamma \hat{u} + \hat{\nu}^{k+1} \mathcal{C}^{(k+1)}\,\widehat{\nabla}\,\hat{u}\Big),$$

which leads to :

$$\mathcal{T}_\delta^\rho \hat{u} = \sum_{k\in\mathbb{Z}} \delta^k \Big(\mathrm{div}_\Gamma + \delta^{-1}\,\widehat{\mathrm{div}}\Big)\Big(\hat{\rho}\hat{\nu}^k \mathcal{C}^{(k)}\,\nabla_\Gamma \hat{u} + \hat{\rho}\hat{\nu}^{k+1} \mathcal{C}^{(k+1)}\,\widehat{\nabla}\,\hat{u}\Big).$$

Therefore a second index rearrangement in this last expression yields:

$$\begin{aligned} \mathcal{T}_\delta^\rho \hat{u} = \sum_{k\in\mathbb{Z}} \delta^k \Big(&\mathrm{div}_\Gamma\left(\hat{\rho}\mathcal{C}^{(k)}\hat{\nu}^k\,\nabla_\Gamma \hat{u}\right) + \widehat{\mathrm{div}}\left(\hat{\nu}^{k+2}\hat{\rho}\mathcal{C}^{(k+2)}\,\widehat{\nabla}\,\hat{u}\right)\Big), \\ + \sum_{k\in\mathbb{Z}} \delta^k \Big(&\mathrm{div}_\Gamma\left(\hat{\nu}^{k+1}\hat{\rho}\mathcal{C}^{(k+1)}\,\nabla_\Gamma \hat{u}\right) + \widehat{\mathrm{div}}\left(\hat{\rho}\mathcal{C}^{(k+1)}\,\nabla_\Gamma \hat{u}\right)\Big), \end{aligned} \tag{2.3.30}$$

Defining for $k \in \mathbb{Z}$ and $\hat{u}' : \Gamma \times \hat{\Omega} \mapsto \mathbb{C}$ the operator $\mathcal{T}^{\rho}_{k+2}$ by:

$$\begin{aligned}\mathcal{T}^{\rho}_{k+2}\hat{u}' := &\operatorname{div}_{\Gamma}\left(\hat{\rho}\mathcal{C}^{(k)}\hat{\nu}^{k}\,\nabla_{\Gamma}\,\hat{u}'\right) + \widehat{\operatorname{div}}\left(\hat{\nu}^{k+2}\hat{\rho}\mathcal{C}^{(k+2)}\,\widehat{\nabla}\,\hat{u}'\right),\\ &+ \operatorname{div}_{\Gamma}\left(\hat{\nu}^{k+1}\hat{\rho}\mathcal{C}^{(k+1)}\,\widehat{\nabla}\,\hat{u}'\right) + \widehat{\operatorname{div}}\left(\hat{\nu}^{k+1}\hat{\rho}\mathcal{C}^{(k+1)}\,\nabla_{\Gamma}\,\hat{u}'\right),\end{aligned}$$

we can rewrite (2.3.30) as follow:

$$\mathcal{T}^{\rho}_{\delta}\hat{u} = \sum_{k\in\mathbb{Z}} \delta^{k-2}\mathcal{T}^{\rho}_{k}\hat{u},$$

which is (2.3.29). Hence by introducing for $k \in \mathbb{Z}$ the operator:

$$\mathcal{T}_k := \mathcal{T}^{\rho}_{k+2} + k^2\mu\hat{\nu}^{k}c^{(k)},$$

we get the expansion (2.3.26).

Now we will use this formula to extend the induction formula (2.3.11) and that will be the object of (2.3.31). By inserting expansion (2.2.7) into the Helmholtz equation (2.1.3) and combining with (2.3.23), we formally get:

$$0 = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \delta^{k+l}\mathcal{T}_k\hat{u}_l = \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} \delta^{k}\mathcal{T}_l\hat{u}_{k-l} = \sum_{k\in\mathbb{Z}} \delta^{k}\left(\sum_{l=0}^{k}\mathcal{T}_l\hat{u}_{k-l}\right).$$

A series of the variable δ is zero for all values of δ if every coefficient is zero. Therefore

$$\sum_{l=0}^{k}\mathcal{T}_l\hat{u}_{k-l} = 0,\ \forall k \geq 0,$$

that can be rewritten as follows:

$$\boxed{\mathcal{T}_0\hat{u}_k = -\sum_{l=1}^{k}\mathcal{T}_l\hat{u}_{k-l},\ \forall k \geq 0.} \tag{2.3.31}$$

Thus, in order to define $\hat{u}_k$ we first need to invert the operator $\mathcal{T}_0$. If a function $u : (x_\Gamma; \hat{x}, \hat{\nu}) \mapsto \mathbb{R}$ does not depend of $\hat{x}$ and $\hat{\nu}$ then this function belong to the kernel of the operator $\mathcal{T}_0$ i.e.

$$\mathcal{T}_0 u = 0.$$

Moreover, we will later see in a functional framework that we have the equivalence:

$$\forall u,\ \mathcal{T}_0 u = 0 \Longleftrightarrow \exists U : \Gamma \mapsto \mathbb{C}\forall(x_\Gamma, \hat{x}, \hat{}) \in \Gamma \times \hat{\Omega} u(x_\Gamma; \hat{x}, \hat{\nu}) = U(x_\Gamma),$$

and this last result is stated in Proposition 2.5.2. Thus the equation (2.3.31) define the near-field $\hat{u}_k$ up to a function which only depends to the variable x_Γ.

2.3.2 Equations for the far-field

Inserting (2.2.8) into (2.1.3) and (2.1.5) yields for all $n \in \mathbb{N}$:

$$\boxed{\operatorname{div}_{\mathcal{L}}\big(\mathcal{C}\,\nabla_{\mathcal{L}}\,u_n\big) + k^2\,\mathrm{C}\,u_n = 0, \quad \text{in } \Omega_0,} \tag{2.3.32}$$

with the boundary condition on $\Gamma \times \{\eta_0\}$:

$$\boxed{\begin{cases} \mathrm{C}\,\partial_\nu u_n = \mathrm{DtN}_{\mathcal{L}}\,u_n \text{ if } n > 0, \\ \mathrm{C}\,\partial_\nu u_0 = \mathrm{DtN}_{\mathcal{L}}\,u_0 + f_{\Sigma_{\eta_0}} \text{ else }. \end{cases}} \tag{2.3.33}$$

We recall that the operator $\mathrm{DtN}_{\mathcal{L}}$ is defined by (1.3.25) and $(\mathcal{C}, C)$ are defined by (1.3.26) and (1.3.24). Nevertheless the boundary conditions on $\Gamma \times \{0\}$ for these problems are missing.

By using the arguments of Proposition 1.3.2,Proposition 1.3.3 and Proposition 1.3.5, we directly have that (2.3.32) and (2.3.33) are equivalent to have for all $n \in \mathbb{N}$ that

$$\boxed{\Delta u^0 + k^2 u^0 = f \quad \text{in } \Omega, \text{ if } n > 0 \text{ then } \Delta u^n + k^2 u^n = 0 \quad \text{in } \Omega,}$$

and the function u^n satisfies the Sommerfeld radiation condition:

$$\boxed{\lim_{R\to\infty} \int_{|x|=R} |\partial_r u^n - iku^n|^2 = 0.}$$

2.3.3 Matching condition

This part is inspired from [34] and [37]. Indeed, we will later prove the existence of a sequence of polynomials $p_n(x_\Gamma, .) \in \mathbb{C}_n[\hat{\nu}]$, where $p_n(x_\Gamma, .) \in \mathbb{C}_n[\hat{\nu}]$ is the set of polynomial functions whose degree is smaller than n, such that the near field has the following expansion uniformly with respect to x_Γ and $\hat{x}$:

$$\boxed{\lim_{\hat{\nu}\to\infty} \hat{u}_n(x_\Gamma; \hat{x}, \hat{\nu}) - p^n(x_\Gamma; \hat{\nu}) = 0.} \tag{2.3.34}$$

Let p_i^n be the i^{th} coefficient of p^n, then replacing $\hat{\nu}$ by $\frac{\nu}{\delta}$ formally yields for $\nu \in [\eta, 2\eta]$ the following expansion for u_δ:

$$u_\delta(x_\Gamma, \nu) \approx \sum_{n=0}^{\infty} \sum_{k=0}^{n} \delta^{n-k} p_k^n(x_\Gamma) \nu^k.$$

Moreover for $\nu \in [\eta, 2\eta]$, using the far field expansion (2.2.8) and Taylor series formally yields the following expansion:

$$u_\delta(x_\Gamma, \nu) \approx \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \delta^n \frac{1}{k!} \partial_\nu^k u^n(x_\Gamma, 0) \nu^k.$$

Then we formally get:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\delta^{n-k}p_k^n(x_\Gamma)\nu^k = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\delta^n\frac{1}{k!}\partial_\nu^k u_n(x_\Gamma,0)\nu^k,$$

and identifying each power of δ and ν of both expansion yields formally the following identity:

$$\boxed{p_k^{n+k}(x_\Gamma) = \tfrac{1}{k!}\partial_\nu^k u_n(x_\Gamma,0).} \tag{2.3.35}$$

In particular one has:

$$p_0^n(x_\Gamma) = u_n(x_\Gamma,0) \quad \text{and} \quad p_1^n(x_\Gamma) = \partial_\nu u_{n-1}(x_\Gamma,0).$$

2.3.4 Summary of the required equations for the ansatz

2.4 Technical assumptions on our physical coefficients

To ensure coercivity properties, we assume that the coefficient $\hat{\rho}$ is bounded from below by a positive constant $\hat{\rho}_c > 0$ i.e.

$$\hat{\rho} \geq \hat{\rho}_c. \tag{2.4.36}$$

For convenience we introduce the infinite strip $\hat{Y}_\infty :=]0,1[^2\times]-1,\infty[$. Moreover we define for $m \leq m_\Gamma$ and any normed vector space $V(\hat{Y}_\infty)$ of function defined on $\hat{Y}_\infty$ the following normed vector spaces: if $m \in \mathbb{N}$ then

$$C^m_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; V(\hat{Y}_\infty)\Big) := \Big\{u \in C^m(\Gamma, V(\hat{Y}_\infty)),\ u \text{ patching-}\psi_\Gamma\text{-admissible}\Big\},$$

and

$$H^m_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; V(\hat{Y}_\infty)\Big) := \Big\{u \in H^m(\Gamma, V(\hat{Y}_\infty)),\ u \text{ patching-}\psi_\Gamma\text{-admissible}\Big\}.$$

Finally we recall that we assumed that our surface Γ is at least $C^{m_\Gamma+1}$ and have the following regularities:

$$(\hat{\rho},\hat{\mu}) \in C^{m_\Gamma}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; L^\infty(\hat{Y}_\infty)\Big) \quad \text{and} \quad \psi_\Gamma \in C^{m_\Gamma+1}\Big(\overline{\Gamma_{\mathrm{M}}}\Big), \tag{2.4.37}$$

where for $m \in \mathbb{N}$, $C^m(\overline{\Gamma_{\mathrm{M}}})$ is the space of restrictions to $\overline{\Gamma_{\mathrm{M}}}$ of functions with C^m regularity on the whole surface Γ.

2.5 Explicit construction of the ansatz

2.5.1 Solution of equation (2.3.31)

2.5.1.1 Functional framework for the strip $\hat{Y}_\infty$

We will see later that for all n and x_Γ our far field $\hat{u}_n(x_\Gamma;\cdot)$ belongs to the space $\mathbb{H}(\hat{Y}_\infty) + \mathbb{C}[\hat{\nu}]$ where we defined the following space:

$$\mathbb{H}(\hat{Y}_\infty) := \left\{u \in H^1_{\mathrm{loc}}(\hat{\Omega}),\ \|u\|^2_{\mathbb{H}(\hat{Y}_\infty)} := \int_{\hat{Y}_\infty}|\nabla u|^2 d\hat{x}d\hat{\nu} + \left|\int_\Sigma u d\hat{x}\right|^2 < \infty \text{ and u is one periodic in } \hat{x}\right\},$$

where $\Sigma :=]0,1[^2 \times \{0\}$. We remark from Poincaré-Wirtinger inequality that $\mathbb{H}(\hat{Y}_\infty)$ is a Hilbert space and for all compact K there exists $C_K > 0$ such that for all $u \in \mathbb{H}(\hat{Y}_\infty)$ the following estimate holds:

$$\|u\|_{L^2(K)} + \|u\|_{L^2(\Sigma)} \leq C_K \|u\|_{\mathbb{H}(\hat{Y}_\infty)}. \tag{2.5.38}$$

Moreover we will see that the right hand-side of (2.3.31) belongs to the space:

$$\mathbb{C}[\hat{\nu}] \oplus \mathbb{H}(\hat{Y}_\infty)^\dagger. \tag{2.5.39}$$

To define this last space we proceed as follow:

1. We introduce the space:

$$\mathbb{H}_{\text{comp}}(\hat{Y}_\infty) := \left\{ \phi \in \mathbb{H}(\hat{Y}_\infty),\ \exists c > 0 \text{ such that } \phi \equiv 0 \text{ on } \mathbb{R}^2 \times]c, \infty[\right\},$$

   and we emphasize that we do not need to provide this last space with topology.

2. We remark that $\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)$ is dense in $\mathbb{H}(\hat{Y}_\infty)$.

3. Thus we can identify the dual space $\mathbb{H}(\hat{Y}_\infty)^\dagger$ as a subset of the dual space $\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger$ with the canonical injection I defined for $(u,v) \in \mathbb{H}(\hat{Y}_\infty)^\dagger \times \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)$ by :

$$\langle Iu, v \rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)} := \langle u, v \rangle_{\hat{Y}_\infty}.$$

   Here we have chosen to compactify the dual bracket $\langle \cdot, \cdot \rangle_{\hat{Y}_\infty} := \langle \cdot, \cdot \rangle_{\mathbb{H}(\hat{Y}_\infty)^\dagger - \mathbb{H}(\hat{Y}_\infty)}$.

4. We identify the space $\mathbb{C}[\hat{\nu}]$ as a subspace of $\mathbb{H}(\hat{Y}_\infty)^\dagger$ with the inclusion map $I : \mathbb{C}[\hat{\nu}] \mapsto \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger$ for $(p, \phi) \in \mathbb{C}[\hat{\nu}] \times \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)$:

$$\langle Iu, v \rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)} := \int_{\hat{Y}_\infty} p(\hat{\nu}) \phi(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}.$$

5. Thus we can write:

$$\mathbb{H}(\hat{Y}_\infty) \cup \mathbb{H}(\hat{Y}_\infty)^\dagger \subset \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger \quad \text{and} \quad \mathbb{C}[\hat{\nu}] \subset \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger,$$

   which give a meaning of (2.5.39).

2.5.1.2 Extension of $\widehat{\text{div}}$ to the space $L^2_{\text{loc}}(\hat{Y}_\infty)$

The operator $\text{div}_{\hat{x},\hat{\nu}}$ is naturally defined as a linear operator $\text{div}_{\hat{x},\hat{\nu}} : C^\infty_{\text{comp}}(\hat{\Omega})^3 \mapsto C^\infty_{\text{comp}}(\hat{\Omega})$ but we will see in the sequel that we need to extend this last operator. Let us explain how we proceed: According to the definition of the space $\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)$ the quantity $\nabla_{\hat{x},\hat{\nu}}\phi$ is well defined and have compact support and we have:

$$\int_{\text{supp}(\phi)} |\nabla_{\hat{x},\hat{\nu}}\phi|^2 d\hat{Y} < \infty. \tag{2.5.40}$$

Let $u \in L^2_{\mathrm{loc}}(\hat{Y}_\infty)$, then since $\mathrm{supp}(\phi)$ is compact we have:

$$\int_{\mathrm{supp}(\phi)} |\boldsymbol{u}|^2 d\hat{Y} < \infty. \tag{2.5.41}$$

Therefore according the Cauchy-Schwarz inequality, (2.5.40) an (2.5.41) leads to:

$$\int_{\mathrm{supp}(\phi)} |\boldsymbol{u} \cdot \nabla_{\hat{x},\hat{\nu}} \phi| d\hat{Y} < \infty.$$

Moreover by using that $\mathrm{supp}(\boldsymbol{u} \cdot \nabla_{\hat{x},\hat{\nu}} \phi) \subset \mathrm{supp}(\phi)$, this imples:

$$\int_{\hat{Y}_\infty} |\boldsymbol{u} \cdot \nabla_{\hat{x},\hat{\nu}} \phi| d\hat{Y} < \infty,$$

and then we now can define the extension of the operator the operator $\mathrm{div}_{\hat{x},\hat{\nu}} : L^2_{\mathrm{loc}}(\hat{Y}_\infty) \mapsto \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)^\dagger$ for $(\boldsymbol{u}, \phi) \in: L^2_{\mathrm{loc}}(\hat{Y}_\infty) \times \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$ by:

$$\langle \mathrm{div}_{\hat{x},\hat{\nu}}(\boldsymbol{u}), \phi \rangle_{\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)^\dagger - \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)} := - \int_{\hat{Y}_\infty} \big(\boldsymbol{u}(\hat{x}, \hat{\nu}), \nabla_{\hat{x},\hat{\nu}} \phi\big) d\hat{x} d\hat{\nu}. \tag{2.5.42}$$

According to the definition of the operator $\widehat{\mathrm{div}}$ (see (2.3.15)), this last operator is naturally defined for $\boldsymbol{u} : \Gamma \mapsto H^1_{\mathrm{loc}}(\hat{Y}_\infty)^3$ by the map $\widehat{\mathrm{div}}\, \boldsymbol{u} : \Gamma \mapsto L^2_{\mathrm{loc}}(\hat{Y}_\infty)$ defined for $x_\Gamma \in \Gamma$ by:

$$\widehat{\mathrm{div}}\, \big(\boldsymbol{u}\big)(x_\Gamma; \cdot) := \mathrm{div}_{\hat{x},\hat{\nu}}\big(M^\dagger(x_\Gamma)\boldsymbol{u}\big),$$

where M is given for $x_\Gamma \in \Gamma_{\mathrm{M}}$ by:

$$M(x_\Gamma) := \big(d\,\psi_\Gamma(x_\Gamma), n(x_\Gamma)\big),$$

and $M(x_\Gamma) := \big(0, n(x_\Gamma)\big)$. By using the same idea, the operator $\widehat{\mathrm{div}}$ is extended for $\boldsymbol{u} : \Gamma \mapsto L^2_{\mathrm{loc}}(\hat{Y}_\infty)$ by $\big(\widehat{\mathrm{div}}\, \boldsymbol{u}\big)(x_\Gamma; \cdot) \in \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)^\dagger$ which is defined for $\phi \in \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$ by:

$$\Big\langle \big(\widehat{\mathrm{div}}\, \boldsymbol{u}\big)(x_\Gamma; \cdot), \phi \Big\rangle_{\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)^\dagger - \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)} := - \int_{\hat{Y}_\infty} \big(\boldsymbol{u}(x_\Gamma; \hat{x}, \hat{\nu}), (\widehat{\nabla}\, \phi)(x_\Gamma; \hat{x}, \hat{\nu})\big) d\hat{x} d\hat{\nu}. \tag{2.5.43}$$

Now let explain why the integral appearing in the righ hand-side of this last definition have a sense: According to the definition of the space $\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$, $\nabla_{\hat{x},\hat{\nu}}\phi$ have a sense. Thus we can give a sense to $M(x_\Gamma)\nabla_{\hat{x},\hat{\nu}}\phi$. Then according to the definition of the operator $\widehat{\nabla}$ (see (2.3.14)) and the one of the matrix $M(x_\Gamma)$ we have:

$$(\widehat{\nabla}\, \phi)(x_\Gamma; \hat{x}, \hat{\nu}) = M(x_\Gamma) \nabla_{\hat{x},\hat{\nu}} \phi.$$

According to the definition of the space $\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$ the quantity $\nabla_{\hat{x},\hat{\nu}}\phi$ is well defined and have compact support and we have:

$$\int_{\mathrm{supp}(\phi)} |\nabla_{\hat{x},\hat{\nu}} \phi|^2 d\hat{Y} < \infty. \tag{2.5.44}$$

Morover, since $\mathrm{supp}(\phi)$ is compact and $u(x_\Gamma;\cdot) \in L^2_{\mathrm{loc}}(\hat{Y}_\infty)$, we have

$$\int_{\mathrm{supp}(\phi)} |\boldsymbol{u}(x_\Gamma;\cdot)|^2 d\hat{Y} < \infty. \tag{2.5.45}$$

Therefore according the Cauchy-Schwarz inequality, (2.5.44) an (2.5.45) leads to:

$$\int_{\mathrm{supp}(\phi)} |\boldsymbol{u}(x_\Gamma;\cdot) \cdot \nabla_{\hat{x},\hat{\nu}}\phi| d\hat{Y} < \infty.$$

Moreover by using that $\mathrm{supp}(\boldsymbol{u}(x_\Gamma;\cdot) \cdot \nabla_{\hat{x},\hat{\nu}}\phi) \subset \mathrm{supp}(\phi)$, this implies:

$$\int_{\hat{Y}_\infty} |\boldsymbol{u}(x_\Gamma;\cdot) \cdot \nabla_{\hat{x},\hat{\nu}}\phi| d\hat{Y} < \infty.$$

2.5.1.3 Construction of a right inverse $\mathcal{T}_0^{-1}$ of the operator $\mathcal{T}_0$

In order to solve (2.3.31), we construct here a right inverse $\mathcal{T}_0^{-1}$ of the operator $\mathcal{T}_0$. That means that this operator should satisfy:

$$\mathcal{T}_0\mathcal{T}_0^{-1} f = f,$$

for all element f of some space specified later. Let us prove the following result to simplify the expression of the operator $\mathcal{T}_0$:

Proposition 2.5.1. *We have that :*

For all $k \in \mathbb{N}$, we have $\mathcal{C}^{(k)} \in C^{m_\Gamma}(\Gamma;\mathcal{L})$ and for all $x_\Gamma \in \Gamma$ we have:

$$\mathcal{C}^{(k)}(x_\Gamma) \cdot T_{x_\Gamma}\Gamma \subset T_{x_\Gamma}\Gamma \quad \text{and} \quad \mathcal{C}^{(k)}(x_\Gamma) \cdot n(x_\Gamma) = c^{(k)} n(x_\Gamma). \tag{2.5.46}$$

- *For $k = 0$, these terms are given by $\mathcal{C}^{(0)} = \mathbb{I}$ and $c^{(0)} = 1$.*
- *For $k = 1$ and all $x_\Gamma \in \Gamma$, the operator $\mathcal{C}^{(1)}(x_\Gamma)$ is the only one such that:*

$$\forall v_\Gamma \in T_{x_\Gamma}\Gamma,\ \mathcal{C}^{(1)}(x_\Gamma) \cdot v_\Gamma = 2 \cdot \big(H(x_\Gamma) - R(x_\Gamma)\big) \cdot v_\Gamma,$$

and $c^{(1)} = 2 \cdot H$.

Proof. Since the idea is the same as the one of [14] we just give the outline of the proof. We first observe that:

$$\mathrm{D}\,\mathcal{L}\,\mathrm{D}\,\mathcal{L}^\dagger = (\mathbb{I} + \nu R)^{-2}. \tag{2.5.47}$$

Combining this last identity with Leibniz formula and the following equalities:

$$\frac{1}{k!}\partial_\nu^k \left(\left(\mathbb{I} + \nu R \right)^{-2} \right) \bigg|_{\nu=0} = (k+1) \cdot (-1)^n R^n \quad \text{and} \quad \det(\mathbb{I} + \nu R) = 1 + 2\nu H + \nu^2 G,$$

directly yields our result. The properties (2.5.46) are direct consequence of (2.5.47) and: $\forall x_\Gamma \in \Gamma,\ R(x_\Gamma) T_{x_\Gamma}\Gamma \subset T_{x_\Gamma}\Gamma$. $\square$

Indeed, thanks to this last result, the operator $\mathcal{T}_0$ is given for $\hat{u} : \Gamma \mapsto P(\mathbb{R}^2)$ and $x_\Gamma \in \Gamma$ by:

$$(\mathcal{T}_0 \hat{u})(x_\Gamma;\cdot) := \mathcal{T}_0(x_\Gamma)\hat{u}(x_\Gamma),$$

where $\mathcal{T}_0(x_\Gamma)$ is defined for $u \in \hat{\Omega} \mapsto \mathbb{C}$ by:

- If $x_\Gamma \in \Gamma_{\mathrm{M}}$:

$$\mathcal{T}_0(x_\Gamma)\hat{u} = \mathrm{div}_{\hat{x},\hat{\nu}}\Big(\rho_{\psi_\Gamma}(x_\Gamma;\cdot)\nabla\hat{u}\Big), \tag{2.5.48}$$

- If $x_\Gamma \notin \Gamma_{\mathrm{M}}$:

$$\mathcal{T}_0(x_\Gamma)\hat{u} := \partial_{\hat{\nu}}\big(\hat{\rho}(x_\Gamma;\cdot)\partial_{\hat{\nu}}\hat{u}(x_\Gamma;\cdot)\big). \tag{2.5.49}$$

Here we defined the map $\rho_{\psi_\Gamma} : \Gamma \times \hat{\Omega} \mapsto \mathcal{L}(\mathbb{R}^3)$ for $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma \times \hat{\Omega}$ by:

$$\rho_{\psi_\Gamma}(x_\Gamma;\hat{x},\hat{\nu}) := \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\begin{pmatrix}\mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger & 0\\ 0 & 1\end{pmatrix}, \tag{2.5.50}$$

and $\partial_{\hat{\nu}}$ is extended for function $v : \hat{\Omega} \mapsto \mathbb{C}$ by the element of $\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)^\dagger$ defined for $\phi \in \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$ by:

$$\langle \partial_{\hat{\nu}} v, \phi\rangle_{\hat{Y}_\infty} = -\int_{\hat{Y}_\infty} v\partial_{\hat{\nu}}\phi d\hat{x}d\hat{\nu}.$$

Let us illustrate this defintion through the example of the unit sphere: In this case we have $\Gamma := \mathbb{S}^2 := \{x \in \mathbb{R}^3,\ |x| = 1\}$ and ψ_Γ is defined for $x \in \Gamma$ by

$$\psi_\Gamma(x_\Gamma) := (\theta,\phi)\,, \tag{2.5.51}$$

where (θ,ϕ) is the unique solution in $[0,2\pi[\times[0,\pi[$ of $x = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$. We have seen in the previous chapter that we can choose:

$$\Gamma_{\mathrm{M}} := \Big\{(\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi), (\theta,\phi) \in [0,2\pi[\times] -\frac{\pi}{2}+\eta, \frac{\pi}{2}+\eta\big[\Big\},$$

and the matrix $\mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger(x_\Gamma)$ is given by:

$$\mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger = \begin{pmatrix}\dfrac{1}{\cos^2(\phi)} & 0\\ 0 & 1\end{pmatrix}.$$

Hence definition (2.5.50) becomes:

$$\rho_{\psi_\Gamma}(x_\Gamma;\hat{x},\hat{\nu}) := \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\begin{pmatrix}\dfrac{1}{\cos^2(\phi)} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

First, we assume that x_Γ belongs to Γ_{M} and we construct now an operator $\mathcal{T}_0(x_\Gamma)$ such that:

$$\mathcal{T}_0(x_\Gamma)\mathcal{T}_0^{-1}(x_\Gamma)f = f. \tag{2.5.52}$$

Hence if $x_\Gamma \in \Gamma_{\mathrm{M}}$ then the operator $\mathcal{T}_0(x_\Gamma)$ is the linear operator associated to the following sesquilinear form:

$$\langle \mathcal{T}_0 u, v\rangle \hat{Y}_\infty = \int_{\hat{Y}_\infty}\hat{\rho}(x_\Gamma;\hat{x}_1,\hat{x}_2,\hat{\nu})\left(\frac{1}{\cos^2(\phi)}\partial_{\hat{x}_1}u\partial_{\hat{x}_1}\overline{v} + \partial_{\hat{x}_2}u\partial_{\hat{x}_2}\overline{v} + \partial_{\hat{\nu}}u\partial_{\hat{\nu}}\overline{v}\right)d\hat{x}d\hat{\nu},$$

and if $x_\Gamma \notin \Gamma_{\mathrm{M}}$ then

$$\langle \mathcal{T}_0 u, v\rangle \hat{Y}_\infty = \int_{\hat{Y}_\infty}\hat{\rho}(x_\Gamma;\hat{x}_1,\hat{x}_2,\hat{\nu})\left(\partial_{\hat{x}_2}\overline{v} + \partial_{\hat{\nu}}u\partial_{\hat{\nu}}\overline{v}\right)d\hat{x}d\hat{\nu}.$$

Proposition 2.5.2. *Let* $u : \Gamma \mapsto \left(\mathbb{H}\big(\hat{Y}_\infty\big) + \mathbb{C}[\hat{\nu}]\right)$ *be patching admissible such that we have:*

$$\mathcal{T}_0 u = 0.$$

Then one has the existence of a function $U : \Gamma \mapsto \mathbb{C}$ *such that :*

$$\forall x_\Gamma \in \Gamma,\ \forall(\hat{x},\hat{\nu}),\ u(x_\Gamma;\hat{x},\hat{\nu}) = U(x_\Gamma).$$

Proof. Let $x_\Gamma \in \Gamma$ and let us prove that $u(x_\Gamma;\cdot)$ is a constant function. By assumption there exist a polynomial $v_2 \in \mathbb{C}[\hat{\nu}]$ and a function $v_1 \in \mathbb{H}\big(\hat{Y}_\infty\big)$ such that:

$$u(x_\Gamma;\cdot) = v_1 + v_2. \tag{2.5.53}$$

We can assume that $v_2(\hat{\nu} = 0) = 0$. Indeed, since spaces $\mathbb{H}\big(\hat{Y}_\infty\big)$ and $\mathbb{C}[\hat{\nu}]$ both contain constant functions then we can replace v_1 and v_2 by $v_1 + v_2(\hat{\nu} = 0)$ and $v_2 - v_2(\hat{\nu} = 0)$ in (2.5.53) respectively:

$$u(x_\Gamma;\cdot) = \underbrace{\big(v_1 + v_2(\hat{\nu} = 0)\big)}_{\in\mathbb{H}\left(\hat{Y}_\infty\right)} + \underbrace{\big(v_2 - v_2(\hat{\nu} = 0)\big)}_{\in\mathbb{C}[\hat{\nu}]}.$$

First we prove that $v_2 = 0$ and finally we prove that v_1 is an constant function.
Proof of $v_2 = 0$:
Since we have $v_2(0) = 0$, one can prove by using the jump formula:

$$\partial_{\hat{\nu}}\mathbb{I}_{\hat{\nu}>0} = \delta \text{ in the sens of distributions}$$

or Proposition 2.5.5 that:

$$\mathcal{T}_0(x_\Gamma)\left(v_2\mathbb{I}_{\hat{\nu}>0}\right) = \mathbb{I}_{\hat{\nu}>0}\frac{\partial^2}{\partial\hat{\nu}^2}v_2 + \frac{\partial}{\partial\hat{\nu}}v_2(0)\delta_\Sigma \quad \text{in} \quad \mathbb{H}_{\text{comp}}\big(\hat{Y}_\infty\big)^\dagger.$$

For convenience, we rewrite this last equality as follows:

$$\mathbb{I}_{\hat{\nu}>0}\frac{\partial^2}{\partial\hat{\nu}^2}v_2 = \mathcal{T}_0(x_\Gamma)\left(v_2\mathbb{I}_{\hat{\nu}>0}\right) - \frac{\partial}{\partial\hat{\nu}}v_2(0)\delta_\Sigma \quad \text{in} \quad \mathbb{H}_{\text{comp}}\big(\hat{Y}_\infty\big)^\dagger. \tag{2.5.54}$$

Thanks to $v_2(0) = 0$, we prove by using the jump formula that $v_2\mathbb{I}_{\hat{\nu}\leq 0} \in \mathbb{H}\big(\hat{Y}_\infty\big)$. Combining this with the continuity of $\mathcal{T}_0(x_\Gamma) : \mathbb{H}\big(\hat{Y}_\infty\big) \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ yields:

$$\mathcal{T}_0(x_\Gamma)\left(v_1 + \mathbb{I}_{\hat{\nu}\leq 0}v_2\right) \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger. \tag{2.5.55}$$

(See (2.5.48) and (2.5.49) for the definition of the operator $\mathcal{T}_0(x_\Gamma)$). Thanks to (2.5.53) we have $u(x_\Gamma;\cdot) = v_1 + v_2\mathbb{I}_{\hat{\nu}\leq 0} + v_2\mathbb{I}_{\hat{\nu}>0}$. Therefore:

$$\mathcal{T}_0(x_\Gamma)\left(v_2\mathbb{I}_{\hat{\nu}>0}\right) = \mathcal{T}_0(x_\Gamma)u(x_\Gamma;\cdot) - \mathcal{T}_0(x_\Gamma)\left(v_1 + v_2\mathbb{I}_{\hat{\nu}\leq 0}\right).$$

Combining this with $\mathcal{T}_0(x_\Gamma)u(x_\Gamma;\cdot) = 0$ and (2.5.55) yields:

$$\mathcal{T}_0(x_\Gamma)\left(v_2\mathbb{I}_{\hat{\nu}>0}\right) \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger.$$

Combining this with $\delta_\Sigma \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$(See (2.5.61) for the definition of δ_Σ) and (2.5.54) leads to:

$$\mathbb{I}_{\hat{\nu}>0} \frac{\partial^2}{\partial \hat{\nu}^2} v_2 \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger. \tag{2.5.56}$$

Since $\mathbb{I}_{\hat{\nu}\leq 0} \frac{\partial^2}{\partial \hat{\nu}^2} v_2$ has a compact support, this last quantity belongs to the space $\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$. Combining this with (2.5.56) and $\frac{\partial^2}{\partial \hat{\nu}^2} v_2 = \mathbb{I}_{\hat{\nu}\leq 0} \frac{\partial^2}{\partial \hat{\nu}^2} v_2 + \mathbb{I}_{\hat{\nu}>0} \frac{\partial^2}{\partial \hat{\nu}^2} v_2$ leads to:

$$\frac{\partial^2}{\partial \hat{\nu}^2} v_2 \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger.$$

Morover since v_2 is a polynomial then the same applies for the function $\dfrac{\partial^2}{\partial \hat{\nu}^2} v_2$ which leads to:

$$\frac{\partial^2}{\partial \hat{\nu}^2} v_2 \in \mathbb{C}[\hat{\nu}] \cap \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger.$$

We will later prove that $\mathbb{C}[\hat{\nu}] \cap \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger = \{0\}$(See Proposition 2.5.9). Thus we deduce that:

$$\frac{\partial^2}{\partial \hat{\nu}^2} v_2 = 0,$$

which combined with $v_2(0) = 0$ leads to:

$$v_2 = \frac{\partial}{\partial \hat{\nu}} v_2(0) \hat{\nu}. \tag{2.5.57}$$

Thus, thanks to $\mathcal{T}_0(x_\Gamma)\big(\mathbb{I}_{\hat{\nu}>0}\hat{\nu}\big) = \delta_\Sigma$(See Proposition 2.5.5) we have:

$$\mathcal{T}_0(x_\Gamma)\left(\mathbb{I}_{\hat{\nu}>0} v_2\right) = \frac{\partial}{\partial \hat{\nu}} v_2(0) \delta_\Sigma.$$

Combining this with $\mathcal{T}_0(x_\Gamma) u(x_\Gamma;\cdot) = 0$ and $v_1 + \mathbb{I}_{\hat{\nu}\leq 0} v_2 = u(x_\Gamma;\cdot) - \mathbb{I}_{\hat{\nu}>0} v_2$ yields:

$$\mathcal{T}_0(x_\Gamma)\big(v_1 + v_2 \mathbb{I}_{\hat{\nu}\leq 0}\hat{\nu}\big) = -\frac{\partial}{\partial \hat{\nu}} v_2(0) \delta_\Sigma. \tag{2.5.58}$$

Moreover by using the jump formula and $v_1 \in \mathbb{H}\big(\hat{Y}_\infty\big)$, we obtains that:

$$v_1 + \frac{\partial}{\partial \hat{\nu}} v_2(0) \hat{\nu} \mathbb{I}_{\hat{\nu}\leq 0} \in \mathbb{H}\big(\hat{Y}_\infty\big).$$

Combining this with the continuity of $\mathcal{T}_0(x_\Gamma) : \mathbb{H}\big(\hat{Y}_\infty\big) \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ yields (See (2.5.48) and (2.5.49) for the definition of the operator $\mathcal{T}_0(x_\Gamma)$):

$$\mathcal{T}_0(x_\Gamma)\left(v_1 + \frac{\partial}{\partial \hat{\nu}} v_2(0) \hat{\nu} \mathbb{I}_{\hat{\nu}\leq 0}\right) \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger.$$

Thus we can apply this last distribution to the test function $1 \in \mathbb{H}\big(\hat{Y}_\infty\big)$. Therefore:

$$\left\langle \mathcal{T}_0(x_\Gamma)\left(v_1 + \frac{\partial}{\partial \hat{\nu}} v_2(0) \hat{\nu} \mathbb{I}_{\hat{\nu}\leq 0}\right), 1 \right\rangle_{\hat{Y}_\infty} = 0.$$

Combining this with $\langle \delta_\Sigma, 1\rangle_{\hat{Y}_\infty} = 1$ and (2.5.58) yields

$$\frac{\partial}{\partial \hat{\nu}} v_2(0) = 0,$$

and then by combining this with (2.5.57), we concludes the proof of $v_2 = 0$.
Proof of v_1 is a constant function:

Thanks to $v_2 = 0$, $\mathcal{T}_0(x_\Gamma)u(x_\Gamma;\cdot) = 0$ and $u(x_\Gamma;\cdot) = v_1 + v_2$, we have $\mathcal{T}_0(x_\Gamma)v_1 = 0$. Morover by assumption, one has $v_1 \in \mathbb{H}(\hat{Y}_\infty)$. Thus we have:

$$\langle \mathcal{T}_0(x_\Gamma)v_1, v_1\rangle_{\hat{Y}_\infty} = -\int_{\hat{Y}_\infty} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})|\widehat{\nabla}\, v_1(x_\Gamma,\hat{x},\hat{\nu})|^2 d\hat{x}d\hat{\nu} = 0.$$

Therefore $\widehat{\nabla}\, v_1(x_\Gamma;\cdot) = 0$. Let us prove that

$$\nabla_{\hat{x}} v_1 = 0. \tag{2.5.59}$$

Assume first that $x_\Gamma \in \Gamma_{\mathrm{M}}$. Then according to the definition of the operator $\widehat{\nabla}$(see (2.3.14)) $\widehat{\nabla}\, v_1(x_\Gamma;\cdot) = 0$ becomes:

$$0 = \mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger \nabla_{\hat{x}} v_1 + \partial_{\hat{\nu}} v_1 \cdot n(x_\Gamma).$$

Projecting this last equality on the tangent space $T_{x_\Gamma}\Gamma$ yields:

$$0 = \mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger \nabla_{\hat{x}} v_1. \tag{2.5.60}$$

Since $x_\Gamma \in \Gamma_{\mathrm{M}}$, one can prove that the linear function $\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger : \mathbb{R}^2 \mapsto T_{x_\Gamma}\Gamma$ is injective. Hence we deduce that $\nabla_{\hat{x}} v_1 = 0$. If $x_\Gamma \notin \Gamma_{\mathrm{M}}$ then using that u is patching admissible directly yields $\nabla_{\hat{x}} v_1 = 0$. Therefore we finished the proof of (2.5.59). According to the definition of the operator $\widehat{\nabla}$ (see (2.3.14)), we deduce by projecting (2.5.60) on $n(x_\Gamma)$ that $\partial_{\hat{\nu}} v_1 = 0$. Combining this with (2.5.59) concludes the proof that v_1 is a constant function. $\square$

First we assume that f is an element of $\mathbb{H}(\hat{Y}_\infty)^\dagger$ in this last equation. Consider the anti-linear form $\delta_\Sigma \in \mathbb{H}(\hat{Y}_\infty)^\dagger$ defined for $\phi \in \mathbb{H}(\hat{Y}_\infty)$ by:

$$\langle \delta_\Sigma, \phi\rangle_{\hat{Y}_\infty} := \int_\Sigma \overline{\phi} d\hat{x}, \tag{2.5.61}$$

and define the operator $\delta_\Sigma \otimes \delta_\Sigma \in \mathcal{L}\left(\mathbb{H}(\hat{Y}_\infty), \mathbb{H}(\hat{Y}_\infty)^\dagger\right)$ defined by:

$$\forall (u,v) \in \mathbb{H}(\hat{Y}_\infty)^2,\ \langle \delta_\Sigma \otimes \delta_\Sigma u, v\rangle_{\hat{Y}_\infty} := \overline{\langle \delta_\Sigma, u\rangle}_{\hat{Y}_\infty} \cdot \langle \delta_\Sigma, v\rangle_{\hat{Y}_\infty}. \tag{2.5.62}$$

Introduce the following constant:

$$C^{\psi_\Gamma}_{\Gamma_{\mathrm{M}}} := \min\left(1, \inf_{x_\Gamma \in \Gamma_{\mathrm{M}}} \left(\inf_{(\hat{x},\hat{\nu}) \in \hat{Y}_\infty} \lambda_{\min}\big(\rho_{\psi_\Gamma}(x_\Gamma;\hat{x},\hat{\nu})\big)\right)\right)$$

where we define for matrix A the quantity $\lambda_{\min}(A)$ as the smallest eigenvalue of A. Thus, thanks to these definitions we can state and prove the following result:

Proposition 2.5.3. *The operator* $-\mathcal{T}_0+\delta_\Sigma\otimes\delta_\Sigma : \mathbb{H}\big(\hat{Y}_\infty\big) \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ *is uniformly coercive on* Γ_M*: For all* $u \in \mathbb{H}\big(\hat{Y}_\infty\big)$*:*

$$\langle -\mathcal{T}_0(x_\Gamma)u + \delta_\Sigma \otimes \delta_\Sigma u, u\rangle_{\hat{Y}_\infty} \geq C^{\psi_\Gamma}_{\Gamma_\mathrm{M}} \|u\|^2_{\mathbb{H}(\hat{Y}_\infty)},$$

and the quantity $C^{\psi_\Gamma}_{\Gamma_\mathrm{M}}$ *is strictly positive.*

Proof. Let $u \in \mathbb{H}\big(\hat{Y}_\infty\big)$. From the definition (2.5.61) and (2.5.62) we have:

$$\langle -\mathcal{T}_0(x_\Gamma)u + \delta_\Sigma \otimes \delta_\Sigma u, u\rangle_{\hat{Y}_\infty} = \langle -\mathcal{T}_0(x_\Gamma)u, u\rangle_{\hat{Y}_\infty} + \big|\langle \delta_\Sigma, u\rangle_{\hat{Y}_\infty}\big|^2 = \langle -\mathcal{T}_0(x_\Gamma)u, u\rangle + \left|\int_\Sigma u d\hat{x}\right|^2.$$

Thanks to (2.5.48) and (2.5.42) this becomes:

$$\begin{aligned}\langle -\mathcal{T}_0(x_\Gamma)u + \delta_\Sigma \otimes \delta_\Sigma u, u\rangle_{\hat{Y}_\infty} &= \int_{\hat{Y}_\infty} \langle \rho_{\psi_\Gamma}(x_\Gamma)\nabla u, \nabla u\rangle d\hat{x}d\hat{\nu} + \left|\int_\Sigma u d\hat{x}\right|^2 \\ &\geq \int_{\hat{Y}_\infty} \lambda_{\min}\big(\rho_{\psi_\Gamma}(x_\Gamma;\hat{x},\hat{\nu})\big)|\nabla u|^2 d\hat{x}d\hat{\nu} + \left|\int_\Sigma u d\hat{x}\right|^2 \geq C^{\psi_\Gamma}_{\Gamma_\mathrm{M}}\|u\|^2_{\mathbb{H}(\hat{Y}_\infty)}.\end{aligned}$$

Thus it remains to prove $\inf_{x_\Gamma\in\Gamma_\mathrm{M}} \lambda_{\min}\big(\rho_{\psi_\Gamma}\big) > 0$. Indeed from (4.5.79) we get that for all $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma \times \hat{Y}_\infty$:

$$\begin{aligned}\lambda_{\min}\big(\rho_{\psi_\Gamma}(x_\Gamma;\hat{x},\hat{\nu})\big) &\geq \hat{\rho}_c \cdot \lambda_{\min}\begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger & 0 \\ 0 & 1\end{pmatrix} \\ &\geq \hat{\rho}_c \min\Big(1, \inf_{x'_\Gamma\in\overline{\Gamma_\mathrm{M}}} \lambda_{\min}\big(\mathrm{D}\,\psi_\Gamma(x'_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x'_\Gamma)^\dagger\big)\Big),\end{aligned}$$

Thus, we now prove that:

$$\inf_{x'_\Gamma\in\overline{\Gamma_\mathrm{M}}} \lambda_{\min}\big(\mathrm{D}\,\psi_\Gamma(x'_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x'_\Gamma)^\dagger\big) > 0.$$

From (2.4.37), $\mathrm{D}\,\psi_\Gamma\,\mathrm{D}\,\psi_\Gamma{}^\dagger$ is a restriction of a continuous function to a compact set $\overline{\Gamma_\mathrm{M}}$. Thus there exists $x_\Gamma \in \overline{\Gamma_\mathrm{M}}$ such that:

$$\inf_{x'_\Gamma\in\overline{\Gamma_\mathrm{M}}} \lambda_{\min}\big(\mathrm{D}\,\psi_\Gamma(x'_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x'_\Gamma)^\dagger\big) = \lambda_{\min}\big(\mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger\big),$$

Since we assumed that $\mathrm{D}\,\psi_\Gamma(x_\Gamma)$ is injective for all $x_\Gamma \in \Gamma_\mathrm{M}$ then this last quantity is strictly positive. □

Thanks to this last result we can deduce that the operator $-\mathcal{T}_0(x_\Gamma) + \delta_\Sigma \otimes \delta_\Sigma : \mathbb{H}\big(\hat{Y}_\infty\big) \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ is invertible and we can solve (2.5.52) with the operator $\big(-\mathcal{T}_0(x_\Gamma) + \delta_\Sigma \otimes \delta_\Sigma\big)^{-1} : \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)$ when $f \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ satisfies the following compatibility condition:

$$\langle f, 1\rangle_{\hat{Y}_\infty} = 0. \tag{2.5.63}$$

To give a sense of this last equality, we emphasize that the function 1 surely belongs to the space $\mathbb{H}\big(\hat{Y}_\infty\big)$.

Proposition 2.5.4. *If $f \in \mathbb{H}(\hat{Y}_\infty)^\dagger$ and satisfies (2.5.63) then:*

$$\mathcal{T}_0(x_\Gamma) \cdot (\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma)^{-1} f = f.$$

Proof. Let $u := (\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma)^{-1} f$, then from (2.5.62) we have:

$$\mathcal{T}_0(x_\Gamma)u - \overline{\langle \delta_\Sigma, u \rangle} \delta_\Sigma = f. \tag{2.5.64}$$

Therefore it remains to prove:

$$\langle \delta_\Sigma, u \rangle_{\hat{Y}_\infty} = 0.$$

Applying (2.5.64) to the test function 1 yields $\langle \mathcal{T}_0(x_\Gamma)u, 1 \rangle_{\hat{Y}_\infty} - \overline{\langle \delta_\Sigma, u \rangle}_{\hat{Y}_\infty} = \langle f, 1 \rangle_{\hat{Y}_\infty}$ and thanks to (2.5.63) this becomes $\overline{\langle \delta_\Sigma, u \rangle}_{\hat{Y}_\infty} = \langle \mathcal{T}_0(x_\Gamma)u, 1 \rangle_{\hat{Y}_\infty}$. Moreover, from the expression (2.5.48) of the operator $\mathcal{T}_0(x_\Gamma)$, we directly get $\langle \mathcal{T}_0(x_\Gamma)u, 1 \rangle_{\hat{Y}_\infty} = 0$ which ends the proof. □

Nevertheless δ_Σ is an element of $\mathbb{H}(\hat{Y}_\infty)^\dagger$ that fails to satisfies (2.5.64) because we have from (2.5.61):

$$\langle \delta_\Sigma, 1 \rangle_{\hat{Y}_\infty} = 1. \tag{2.5.65}$$

However we have the following results:

Proposition 2.5.5. *For all $x_\Gamma \in \Gamma$ we have in the space $\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger$:*

$$\mathcal{T}_0(x_\Gamma)\hat{\nu}_+ = \delta_\Sigma,$$

where $\hat{\nu}_+ = 0$ if $\hat{\nu} \le 0$ and $\hat{\nu}_+ = \hat{\nu}$ if not.

Proof. We only give the proof for $x_\Gamma \in \Gamma_{\mathrm{M}}$ because the one in the contrary case is similar. Let $(\hat{x}, \hat{\nu}) \mapsto \phi(\hat{x}, \hat{\nu}) \in \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)$. Applying the definitions (2.5.48) and (2.5.42) yields:

$$\begin{aligned}
\langle \mathcal{T}_0(x_\Gamma)\hat{\nu}_+, \phi \rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}(\hat{Y}_\infty)} &= -\langle \rho_{\psi_\Gamma}(x_\Gamma) \nabla \hat{\nu}_+, \nabla \phi \rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}(\hat{Y}_\infty)}, \\
&= -\int_{\hat{Y}_+} \partial_\nu \phi \, d\hat{x} d\hat{\nu} = \int_\Sigma \phi \, d\hat{x},
\end{aligned}$$

which ends the proof. □

Corollary 2.5.6. *If $f \in \mathbb{H}(\hat{Y}_\infty)^\dagger$ and define:*

$$\mathcal{T}_0^{-1}(x_\Gamma) f := \langle f, 1 \rangle_{\hat{Y}_\infty} \hat{\nu}_+ + (\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma)^{-1} \big(f - \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma \big), \tag{2.5.66}$$

then (2.5.52) holds and $\mathcal{T}_0^{-1}(x_\Gamma) f \in \mathbb{H}(\hat{Y}_\infty) + \mathbb{C}\hat{\nu}$.

Proof. The following decomposition holds:

$$f = \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma + \big(f - \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma \big). \tag{2.5.67}$$

Thanks to (2.5.65), we have that $\big(f - \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma \big)$ satisfies (2.5.63). Thus we can apply Proposition 2.5.4 which yields:

$$\mathcal{T}_0(x_\Gamma)(\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma)^{-1} \big(f - \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma \big) = f - \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma. \tag{2.5.68}$$

Moreover, from Proposition 2.5.5 we have $\mathcal{T}_0\big(\langle f, 1 \rangle_{\hat{Y}_\infty} \hat{\nu}_+ \big) = \langle f, 1 \rangle_{\hat{Y}_\infty} \delta_\Sigma$. Combining this with (2.5.68) and (2.5.67) conclude the proof. □

Now, we assume that f is a polynomial of degree $n \in \mathbb{N}$. In the case of $\hat{\rho} = 1$, the operator $d_{\hat{\nu}}^{-2} : \mathbb{C}_n[\hat{\nu}] \mapsto \mathbb{C}_{n+2}[\hat{\nu}]$ defined for $p \in \mathbb{C}_n[\hat{\nu}]$ by $d_{\hat{\nu}}^{-2}p := u$ by the unique solution u of:

$$\frac{d^2u}{d\hat{\nu}^2} = p, \quad u(0) = 0 \quad \text{and} \quad \frac{du}{d\hat{\nu}}(-1) = 0, \tag{2.5.69}$$

satisfies for all $p \in \mathbb{C}_n[\hat{\nu}]$, $\mathcal{T}_0 d_{\hat{\nu}}^{-2}(x_\Gamma)p = p$. Nevertheless when $\hat{\rho}$ fails to be equal to the constant one we could have:

$$\mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}f - f \neq 0.$$

However, we have the following result:

Proposition 2.5.7. *For all $p \in \mathbb{C}_n[\hat{\nu}]$ we have:*

$$\mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \quad \textit{and} \quad \big\langle \mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p, 1\big\rangle_{\hat{Y}_\infty} = 0.$$

Moreover $\mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p$ is given for $\phi \in \mathbb{H}_{\text{comp}}\big(\hat{Y}_\infty\big)$ by:

$$\begin{aligned}
\langle \mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p, \phi\rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)} &= -\int_{\hat{Y}_-} p(\hat{\nu}) \cdot \overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} \\
&\quad - \int_{\hat{Y}_-} \hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{\hat{\nu}}(d_{\hat{\nu}}^{-2}p)(\hat{\nu}) \cdot \partial_{\hat{\nu}}\overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}, \\
&\quad + \int_{\Sigma} \partial_{\hat{\nu}}(d_{\hat{\nu}}^{-2}p)(\hat{\nu}) \cdot \overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x},
\end{aligned} \tag{2.5.70}$$

where $\hat{Y}_- :=]0,1[^2\times]-1,0[$ and $\hat{Y}_+ :=]0,1[^2\times]0,\infty[$.

Proof. Let $\phi \in \mathbb{H}_{\text{comp}}\big(\hat{Y}_\infty\big)$ and let us prove (2.5.70). Applying (2.5.48) and (2.5.42) yields:

$$\begin{aligned}
\langle \mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p, \phi\rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)} &= -\int_{\hat{Y}_\infty} \hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{\hat{\nu}}(d_{\hat{\nu}}^{-2}p)(\hat{\nu}) \partial_{\hat{\nu}}\overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} \\
&\quad - \int_{\hat{Y}_\infty} p(\hat{\nu}) \cdot \overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu},
\end{aligned}$$

and splitting these two integrals yields:

$$\begin{aligned}
\langle \mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p, \phi\rangle_{\mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger - \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)} &= -\int_{\hat{Y}_-} p(\hat{\nu}) \cdot \overline{\phi}(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} \\
&\quad - \int_{\hat{Y}_-} \hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{\hat{\nu}}\big(d_{\hat{\nu}}^{-2}p\big)(\hat{\nu}) \cdot \partial_{\hat{\nu}}\overline{\phi} d\hat{x} d\hat{\nu}, \\
&\quad - \int_{\hat{Y}_+} \partial_{\hat{\nu}}\big(d_{\hat{\nu}}^{-2}p\big)(\hat{\nu}) \partial_{\hat{\nu}}\overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} - \int_{\hat{Y}_+} p(\hat{x}, \hat{\nu}) \overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}.
\end{aligned} \tag{2.5.71}$$

Moreover, doing an integration by parts and using (2.5.69), yields:

$$\begin{aligned}
\int_{\hat{Y}_+} \partial_{\hat{\nu}}\big(d_{\hat{\nu}}^{-2}p\big)(\hat{\nu}) \partial_{\hat{\nu}}\overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} + \int_{\hat{Y}_+} p(\hat{x}, \hat{\nu}) \overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} &= -\cancel{\int_{\hat{Y}_+} \big(\partial_{\hat{\nu}}^2\big(d_{\hat{\nu}}^{-2}p\big)(\hat{\nu}) - p(\hat{\nu})\big)\overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}}, \\
&\quad - \int_{\Sigma} \partial_{\hat{\nu}} d_{\hat{\nu}}^{-2}p \cdot \overline{\phi}(\hat{x}, \hat{\nu}) d\hat{x} = 0,
\end{aligned}$$

which concludes the proof of (2.5.70) by combination with (2.5.71). □

Corollary 2.5.8. *If $f \in \mathbb{C}_n[\hat{\nu}]$ and define:*

$$\mathcal{T}_0^{-1}(x_\Gamma)f = \left(\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma\right)^{-1} \left(\mathcal{T}_0(x_\Gamma) d_{\hat{\nu}}^{-2} f - f\right) + d_{\hat{\nu}}^{-2} f, \tag{2.5.72}$$

then (2.5.52) *holds and* $\mathcal{T}_0^{-1}(x_\Gamma)f \in \mathbb{H}(\hat{Y}_\infty) + \mathbb{C}_{n+2}[\hat{\nu}]$.

Proof. Using Proposition 2.5.7 and Proposition 2.5.4 yields:

$$\mathcal{T}_0(x_\Gamma) \left(\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma\right)^{-1} \left(\mathcal{T}_0(x_\Gamma) d_{\hat{\nu}}^{-2} f - f\right) = \mathcal{T}_0(x_\Gamma) d_{\hat{\nu}}^{-2} f - f,$$

which directly conclude the proof of (2.5.52). $\mathcal{T}_0^{-1}(x_\Gamma)f \in \mathbb{H}(\hat{Y}_\infty) + \mathbb{C}_{n+2}[\hat{\nu}]$ is a direct consequence of the definition of $d_{\hat{\nu}}^{-2} f \in \mathbb{C}_{n+2}[\hat{\nu}]$ and $\left(\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma\right)^{-1} \left(\mathcal{T}_0(x_\Gamma) d_{\hat{\nu}}^{-2} f - f\right) \in \mathbb{H}(\hat{Y}_\infty)$. □

Now, we assume that f takes the forms:

$$f = f_1 + f_2, \tag{2.5.73}$$

for some $(f_1, f_2) \in \mathbb{H}(\hat{Y}_\infty)^\dagger \times \mathbb{C}_n[\hat{\nu}]$. This decomposition is unique because we have the following result:

Proposition 2.5.9. *The space $\mathbb{H}(\hat{Y}_\infty)^\dagger$ and $\mathbb{C}[\hat{\nu}]$ satisfies $\mathbb{H}(\hat{Y}_\infty)^\dagger \cap \mathbb{C}[\hat{\nu}] = \{0\}$.*

Proof. Define the sequence of function (1_n) defined for $\hat{\nu} \in]-1, \infty[$ by $1_n(\hat{\nu}) = \chi(\hat{\nu} - n)$ where χ is a regular function such that $\chi \equiv 1$ on $]-\infty, 0[$ and $\chi \equiv 0$ on $]1, \infty[$. Since for all $n \leq 0$ the function 1_n and its gradient has compact support, we have that $1_n \in \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)$ for all n and we have for large n

$$\|1_n\|_{\mathbb{H}(\hat{Y}_\infty)} = \sqrt{1 + \int_0^1 |\chi'(s)|^2 ds}. \tag{2.5.74}$$

First, we prove for all $k \in \mathbb{N}$ the following equivalence:

$$\langle \hat{\nu}^k, 1_n \rangle_{\hat{Y}_\infty} = \int_{\hat{Y}_\infty} \hat{\nu}^k 1_n d\hat{x} d\hat{\nu} \underset{n\to\infty}{\sim} \frac{n^{k+1}}{k+1}. \tag{2.5.75}$$

Indeed on the one hand we have:

$$\int_{\hat{Y}_\infty} \hat{\nu}^k 1_n d\hat{x} d\hat{\nu} = \int_{-1}^{n} s^k ds + \int_n^{n+1} s^k 1_n ds = \frac{n^{k+1} - (-1)^{k+1}}{k+1} + \int_n^{n+1} s^k 1_n ds.$$

On the other hand we have:

$$\left| \int_n^{n+1} s^k 1_n ds \right| \leq \int_n^{n+1} s^k ds = \frac{1}{k+1} \sum_{q=0}^{k} \binom{k+1}{q} n^q = \underset{n\to\infty}{o}(n^{k+1}).$$

Let $p \in \mathbb{H}(\hat{Y}_\infty)^\dagger \cap \mathbb{C}[\hat{\nu}]$. Since from (2.5.74) the sequence 1_n is bounded in the space $\mathbb{H}(\hat{Y}_\infty)$ thus we should have:

$$\sup_{n\in\mathbb{N}} |\langle p, 1_n \rangle_{\hat{Y}_\infty}| < \infty. \tag{2.5.76}$$

Assume by contradiction that $p \neq 0$. Let k be the degrees of p. By the definition of the degree we have $p_k \neq 0$. Nevertheless the equivalence (2.5.75) implies that:

$$\langle p, 1_n \rangle_{\hat{Y}_\infty} \underset{n\to\infty}{\sim} \frac{n^{k+1}}{k+1} p_k,$$

which contradicts (2.5.76). □

Thanks to the last result we can write $\mathbb{H}(\hat{Y}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}]$ and define $\mathcal{T}_0^{-1}(x_\Gamma)f$ by:

$$\mathcal{T}_0^{-1}(x_\Gamma)f := \mathcal{T}_0^{-1}(x_\Gamma)f_1 + \mathcal{T}_0^{-1}(x_\Gamma)f_2,$$

where we recall from (2.5.66) and (2.5.72) that:

$$\begin{cases} \mathcal{T}_0^{-1}(x_\Gamma)f_1 := \langle f_1, 1\rangle_{\hat{Y}_\infty}\hat{\nu}_+ + (\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma)^{-1}\big(f_1 - \langle f_1, 1\rangle_{\hat{Y}_\infty}\delta_\Sigma\big), \\ \mathcal{T}_0^{-1}(x_\Gamma)f_2 := (\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma)^{-1}\left(\mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}f_2 - f_2\right) + d_{\hat{\nu}}^{-2}f_2. \end{cases}$$

Therefore thanks to Corollary 2.5.6, Corollary 2.5.8 if $x_\Gamma \in \Gamma_{\mathrm{M}}$ then we have $\mathcal{T}_0^{-1}(x_\Gamma) : \mathbb{H}(\hat{Y}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}] \mapsto \mathbb{H}(\hat{Y}_\infty) + \mathbb{C}[\hat{\nu}]$ and (2.5.52) holds.

Now consider $x_\Gamma \notin \Gamma_{\mathrm{M}}$ and let us study the invertibility of $\mathcal{T}_0(x_\Gamma)$. In this case we emphasize that for all $n \in \mathbb{N}$ the near field $\hat{u}_n(x_\Gamma;\cdot)$ and coefficient appearing in the definition of the operator $\mathcal{T}_0(x_\Gamma)$ are independent of the variable $\hat{x}$. We identify the set of functions defined on $\hat{Y}_\infty$ independent of the variable $\hat{x}$ with the set of function defined on $\hat{I}_\infty :=]-1,\infty[$ with the following inclusion map:

$$\left(\hat{\nu} \in \hat{I}_\infty \mapsto f(\hat{\nu})\right) \mapsto \left((\hat{x},\hat{\nu}) \in \hat{Y}_\infty \mapsto f(\hat{\nu})\right).$$

We introduce the following space:

$$\mathbb{H}(\hat{I}_\infty) := \left\{u \in H^1_{\mathrm{loc}}(\hat{I}_\infty),\ \|u\|^2_{\mathbb{H}(\hat{I}_\infty)} := |u(0)|^2 + \int_{\hat{I}_\infty} |\partial_{\hat{\nu}}u|^2 \mathrm{d}\hat{\nu} < \infty\right\}.$$

Let $\delta_0 \in \mathbb{H}(\hat{I}_\infty)^\dagger$ be defined for $\phi \in \mathbb{H}(\hat{I}_\infty)$ by:

$$\langle \delta_0, \phi\rangle_{\hat{I}_\infty} := \overline{\phi(0)},$$

where $\langle\cdot,\cdot\rangle_{\hat{I}_\infty}$ is the dual bracket in $\mathbb{H}(\hat{I}_\infty)$. We define $\delta_0 \otimes \delta_0 \in \mathcal{L}\left(\mathbb{H}(\hat{I}_\infty), \mathbb{H}(\hat{I}_\infty)^\dagger\right)$ defined for $u \in \mathbb{H}(\hat{Y}_\infty)$ by:

$$\delta_0 \otimes \delta_0 \cdot u := u(0)\delta_0.$$

Then from (4.5.79) it is trivial that the operator $\mathcal{T}_0(x_\Gamma) - \delta_0 \otimes \delta_0 : \mathbb{H}(\hat{I}_\infty) \mapsto \mathbb{H}(\hat{I}_\infty)^\dagger$ is coercive with coercivity constant $\hat{\rho}_c$. Thus in the same way than for $x_\Gamma \in \Gamma_{\mathrm{M}}$ we can define the operator $\mathcal{T}_0^{-1}(x_\Gamma)$ for $f \in \mathbb{H}(\hat{I}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}]$ by:

$$\mathcal{T}_0^{-1}(x_\Gamma)f := (\mathcal{T}_0 - \delta_0 \otimes \delta_0)^{-1}\left(f_1 - \langle f_1, 1\rangle_{\hat{I}_\infty}\delta_0 - \mathcal{T}_0(x_\Gamma)d_{\hat{\nu}}^{-2}f_2 + f_2\right) + \langle f_1, 1\rangle_{\hat{I}_\infty}\hat{\nu}_+ + d_{\hat{\nu}}^{-2}f_2, \tag{2.5.77}$$

with $f = f_1 + f_2$ and $(f_1, f_2) \in \mathbb{H}(\hat{I}_\infty)^\dagger \times \mathbb{C}[\hat{\nu}]$. Following the same way we easily get the following result:

Proposition 2.5.10. *For all $x_\Gamma \notin \Gamma_{\mathrm{M}}$ the operator $\mathcal{T}_0^{-1}(x_\Gamma) : \mathbb{H}(\hat{I}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}] \mapsto \mathbb{H}(\hat{I}_\infty) + \mathbb{C}[\hat{\nu}]$ defined by (2.5.77) is a right inverse of the operator $\mathcal{T}_0(x_\Gamma)$ i.e.*

$$\forall f \in \mathbb{H}(\hat{I}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}],\ \mathcal{T}_0(x_\Gamma)\mathcal{T}_0^{-1}(x_\Gamma)f = f.$$

Now let us investigate regularity of $x_\Gamma \mapsto \mathcal{T}_0^{-1}(x_\Gamma)f(x_\Gamma)$ for a given f. For convenience we introduce the application $\mu\Big(.\Big)$ defined for $f \in \mathbb{C}[\hat{\nu}] \oplus \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ by:

$$\mu\Big(f\Big) := \int_0^1 f_1(\hat{\nu})d\hat{\nu} + \langle f_2, 1\rangle_{\hat{Y}_\infty} \quad \text{with} \quad (f_1, f_2) \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \times \mathbb{C}[\hat{\nu}]. \tag{2.5.78}$$

Proposition 2.5.11. *Let $m \leq m_\Gamma$, $d \in \mathbb{N}$ and $f := f_1+f_2$ with $(f_1, f_2) \in H^m_{0,\Gamma_\mathrm{M}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big)\times H^m\Big(\Gamma; \mathbb{C}_d[\hat{\nu}]\Big)$. We have the following regularity:*

$$u := \mathcal{T}_0^{-1} f \in H^m\Big(\Gamma; \mathbb{C}_{d+2}[\hat{\nu}]\Big) + H^m_{0,\Gamma_\mathrm{M}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big). \tag{2.5.79}$$

Moreover $\mathcal{T}_0^{-1}u$ takes the form $\mathcal{T}_0^{-1} = u_1 + u_2$ with $u_1 \in H^m_{0,\Gamma_\mathrm{M}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big)$ and for all $x_\Gamma \in \Gamma$ the polynomial $u_2(x_\Gamma; \cdot)$ is the unique solution of:

$$\frac{d^2u_2(x_\Gamma; \cdot)}{d\hat{\nu}^2} = f_2(x_\Gamma; \cdot), \quad u_2(x_\Gamma; 0) = 0 \quad \textit{and} \quad \frac{du_2(x_\Gamma; 0)}{d\hat{\nu}} = \mu\Big(f(x_\Gamma; \cdot)\Big). \tag{2.5.80}$$

Before the proof of this proposition we prove an intermediate result.

Proposition 2.5.12. *We can extend the application $G := \mathrm{D}\,\psi_\Gamma \,\mathrm{D}\,\psi_\Gamma{}^\dagger : \Gamma_\mathrm{M} \mapsto \mathbb{R}^2$ to an application $\tilde{G} \in C^{m_\Gamma}(\Gamma; \mathcal{L}(\mathbb{R}^2))$ satisfying:*

$$\inf_{x_\Gamma \in \Gamma} \lambda_{\min}\Big(\tilde{G}(x_\Gamma)\Big) \geq \frac{G_{\min}}{2}, \tag{2.5.81}$$

where $G_{\min}$ is defined by $G_{\min} := \inf_{x_\Gamma \in \Gamma_\mathrm{M}} \lambda_{\min}(G(x_\Gamma))$.

Proof. From (2.4.37) we deduce that G is the restriction to Γ_M of some $\tilde{G}' \in C^{m_\Gamma}\big(\Gamma, \mathcal{L}(\mathbb{R}^2)\big)$. Thus we can introduce the following open subset of Γ:

$$L_\Gamma := \left\{x_\Gamma \in \Gamma, \lambda_{\min}\big(\tilde{G}'(x_\Gamma)\big) < \frac{G_{\min}}{2}\right\}.$$

First, we prove existence of a function $\phi_I \in C^{m_\Gamma}(\Gamma; [0,1])$ satisfying:

$$\phi_I \equiv 1 \quad \text{on} \quad L_\Gamma \quad \text{and} \quad \phi_I \equiv 0 \quad \text{on} \quad \Gamma_\mathrm{M}. \tag{2.5.82}$$

Let us prove by contradiction that:

$$\eta := \mathrm{dist}(\Gamma_\mathrm{M}, L_\Gamma) > 0. \tag{2.5.83}$$

Let $(x_k^0)_k$ and $(x_k^0)_k$ be sequence of Γ_M and L_Γ such that:

$$\lim |x_k^0 - x_k^1| = 0. \tag{2.5.84}$$

Since $\overline{\Gamma_\mathrm{M}}$ and $\overline{L_\Gamma}$ are compact there exist $(x_0, x_1) \in \overline{\Gamma_\mathrm{M}} \times \overline{L_\Gamma}$ such that $(x_k^0)_k$ and $(x_k^1)_k$ respectively converge up to a sub-sequence. Thanks to (2.5.84) we have $x_0 = x_1$ which leads to the existence of $x \in \overline{L_\Gamma} \cap \overline{\Gamma_\mathrm{M}}$. We can prove that the map $x'_\Gamma \in \Gamma \mapsto \lambda_{\min}\big(\tilde{G}'(x'_\Gamma)\big)$ is smooth. Therefore from the definition of $G_{\min}$ and the set L_Γ we have:

$$\lambda_{\min}(\tilde{G}'(x_\Gamma)) \geq G_{\min} \quad \text{and} \quad \lambda_{\min}(\tilde{G}'(x_\Gamma)) \leq \frac{G_{\min}}{2},$$

which is contradictory. From the definition of L_Γ we have: $L_\Gamma \cap \Gamma_{\mathrm{M}} = \emptyset$. Let $(\phi_i)_{i\in\mathbb{N}}$ be a partition of unity of $\mathbb{R}^3$ with C^∞ regularity such that for all $i \in \mathbb{N}$ and $(x, y) \in \operatorname{supp}(\phi_i)$ we have:

$$|x - y| \leq \frac{\eta}{2}. \tag{2.5.85}$$

The existence of $(\phi_i)_{i\in\mathbb{N}}$ is a direct consequence of [57, Theorem 3.21]. Define the following set of indexes

$$I := \{i \in \mathbb{N},\ \operatorname{supp}(\phi_i) \cap L_\Gamma \neq \emptyset\},$$

and choose the function ϕ_I as the restriction on Γ of the function:

$$\phi_I := \sum_{i\in I} \phi_i. \tag{2.5.86}$$

Let us show that for all $x_\Gamma \in \Gamma_{\mathrm{M}}$ we have $\phi_L(x_\Gamma) = 0$. Assume that $\phi_L(x_\Gamma) \neq 0$. Then there exists $i \in I$ such that $x_\Gamma \in \operatorname{supp}(\phi_i)$. From the definition of the set I there exists $y \in \operatorname{supp}(\phi) \cap L_\Gamma$. Since x_Γ and y both belong to $\operatorname{supp}(\phi_i)$, we have from (2.5.85):

$$|x_\Gamma - y| \leq \frac{\eta}{2}.$$

Therefore from $x_\Gamma \in \Gamma_{\mathrm{M}}$ and $y \in L_\Gamma$, we have $\operatorname{dist}(\Gamma_{\mathrm{M}}, L_\Gamma) \leq \frac{\eta}{2}$ which contradict (2.5.83). Now let us prove that for all $x_\Gamma \in L_\Gamma$ we have $\phi_I(x_\Gamma) = 1$. Since $(\phi_i)_{i\in\mathbb{N}}$ is a partition of unity we have

$$\sum_{i\in\mathbb{N}} \phi_i(x_\Gamma) = 1$$

and then from (2.5.86) it is sufficient to prove that:

$$\forall i \notin I,\ \ \phi_i(x_\Gamma) = 0.$$

Indeed, we have from the definition of I for all $i \notin I$ $\operatorname{supp}(\phi_i) \cap L_\Gamma = \emptyset$. Therefore Since $x_\Gamma \in L_\Gamma$ we have $x_\Gamma \notin \phi_i(x_\Gamma)$ which conclude the proof of (2.5.82).

Moreover since Γ is a C^{m_Γ} manifolds and for all i, ϕ_i is a C^∞ function on $\mathbb{R}^3$ then the function ϕ_I belongs to $C^{m_\Gamma}(\Gamma)$.

Thanks to the function ϕ_I, we now can define the map $\tilde{G}$ by:

$$\tilde{G} := (1 - \phi_I) \cdot \tilde{G}' + \phi_I \cdot \frac{G_{\min}}{2}\mathbb{I}$$

and prove that this last function satisfies the desired property.

The regularity of ϕ and $\tilde{G}'$ implies that $\tilde{G}$ belongs to $C^{m_\Gamma}(\Gamma, \mathcal{L}(\mathbb{R}^2))$. By construction, the function ϕ_I satisfies $\phi_I \equiv 0$ on Γ_{M} which implies that we have well $G \equiv \tilde{G}$ on Γ_{M}.

Finally it remains to prove (2.5.81). Indeed let $x_\Gamma \in \Gamma$ if $x_\Gamma \notin L_\Gamma$ then by definition of L_Γ we have:

$$\lambda_{\min}(\tilde{G}'(x_\Gamma)) \geq \frac{G_{\min}}{2}$$

and combining with $0 \leq \phi_I \leq 1$ yields:

$$\lambda_{\min}\left(\tilde{G}(x_\Gamma)\right) = (1 - \phi_I(x_\Gamma)) \cdot \lambda_{\min}\left(\tilde{G}'(x_\Gamma)\right) + \phi_E(x_\Gamma) \cdot \frac{G_{\min}}{2} \geq \frac{G_{\min}}{2}. \tag{2.5.87}$$

If $x_\Gamma \in L_\Gamma$ we have $\phi_I(x_\Gamma) = 1$ which leads to:

$$\tilde{G}(x_\Gamma) = \frac{G_{\min}}{2}\mathbb{I},$$

and this matrix clearly satisfies (2.5.87). □

We also need a second intermediate result. It is easy to adapt the proof of [57, Theorem 3.20] to get the following generalization:

Proposition 2.5.13. *Let $m \leq m_\Gamma$ and two Hilbert spaces E, F. Then:*

$$\forall (A,u) \in C^{m_\Gamma}\big(\Gamma; \mathcal{L}(E,F)\big) \times H^m\big(\Gamma; E\big), \;\; Au \in H^m(\Gamma, F),$$

where $Au : \Gamma \mapsto F$ is the map defined for x_Γ by $A(x_\Gamma)u(x_\Gamma)$. Moreover there exists $C_m > 0$ independent from A and K such that:

$$\|Au\|_{H^m(\Gamma,F)} \leq C_m \|A\|_{C^{m_\Gamma}\left(\Gamma;\mathcal{L}(E,F)\right)} \cdot \|u\|_{H^m(\Gamma,E)}.$$

Proof of Proposition 2.5.11. We recall that $u := \mathcal{T}_0^{-1} f$ is defined by $u := u_1 + u_2$ where:

- The quantity $u_1(x_\Gamma;\cdot)$ is given for for all $x_\Gamma \in \Gamma_{\mathrm{M}}$ by:

$$\begin{aligned} u_1(x_\Gamma;\cdot) := & \left(\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma\right)^{-1} \Big(f_1(x_\Gamma;\cdot) - \langle f_1(x_\Gamma;\cdot), 1\rangle_{\hat{Y}_\infty} \delta_\Sigma\Big), \\ & + \left(\mathcal{T}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma\right)^{-1} \Big(\mathcal{T}_0(x_\Gamma) d_{\hat{\nu}}^{-2} f_2(x_\Gamma;\cdot) - f_2(x_\Gamma;\cdot)\Big), \\ & + \langle f_1(x_\Gamma;\cdot), 1\rangle_{\hat{Y}_\infty} (\hat{\nu}_+ - \hat{\nu}), \end{aligned} \tag{2.5.88}$$

  and for all $x_\Gamma \in \Gamma \setminus \Gamma_{\mathrm{M}}$:

$$\begin{aligned} u_1(x_\Gamma;\cdot) := & \left(\mathcal{T}_0(x_\Gamma) - \delta_0 \otimes \delta_0\right)^{-1} \Big(f_1(x_\Gamma;\cdot) - \langle f_1(x_\Gamma;\cdot), 1\rangle_{\hat{I}_\infty} \delta_0\Big), \\ & + \left(\mathcal{T}_0(x_\Gamma) - \delta_0 \otimes \delta_0\right)^{-1} \Big(\mathcal{T}_0(x_\Gamma) d_{\hat{\nu}}^{-2} f_2(x_\Gamma;\cdot) - f_2(x_\Gamma;\cdot)\Big), \\ & + \langle f_1(x_\Gamma;\cdot), 1\rangle_{\hat{Y}_\infty} (\hat{\nu}_+ - \hat{\nu}). \end{aligned} \tag{2.5.89}$$

- The quantity $u_2(x_\Gamma;\cdot)$ is given for $x_\Gamma \in \Gamma$ by:

$$u_2(x_\Gamma;\cdot) := \langle f_1(x_\Gamma;\cdot), 1\rangle_{\hat{Y}_\infty} \hat{\nu} + d_{\hat{\nu}}^{-2} f_2(x_\Gamma;\cdot). \tag{2.5.90}$$

First, from the expression (2.5.90) of u_2 , the property (2.5.69) satisfied by the operator $d_{\hat{\nu}}^{-2}$ and the definition of μ given by (2.5.78) we directly conclude the proof of (2.5.80). Moreover, thanks to $(f_1, f_2) \in H^m_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big) \times H^m\Big(\Gamma; \mathbb{C}_d[\hat{\nu}]\Big)$ we have from (2.5.90) that $u_2 \in H^m(\Gamma; \mathbb{C}_{n+2}[\hat{\nu}])$. Thus it remains to prove $u_1 \in H^m_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big)$.

We emphasize that the map $x_\Gamma \mapsto \mathcal{T}_0^{-1}(x_\Gamma)$ is piece-wise defined by (2.5.48) and (2.5.49) and then this complicates the proof of regularity of the map $x_\Gamma \in \Gamma \mapsto u_1(x_\Gamma;\cdot)$ on the interface $\partial\Gamma_{\mathrm{M}}$. Therefore, we introduce the map:

$$\tilde{\mathcal{T}}_0 : \Gamma \mapsto \mathcal{L}\left(\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big) + \mathbb{C}[\hat{\nu}], \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)^\dagger \oplus \mathbb{C}[\hat{\nu}]\right),$$

defined for $x_\Gamma \in \Gamma$ as the linear operator $\tilde{\mathcal{T}}_0(x_\Gamma)$. This operator is defined for $v \in \mathbb{H}_{\text{comp}}\big(\hat{Y}_\infty\big) + \mathbb{C}[\hat{\nu}]$ by:

$$\tilde{\mathcal{T}}_0(x_\Gamma)v := \operatorname{div}_{\hat{x},\hat{\nu}}\left(\hat{\rho}(x_\Gamma;\cdot)\begin{pmatrix}\tilde{G}(x_\Gamma) & 0\\ 0 & 1\end{pmatrix}\nabla_{\hat{x},\hat{\nu}}v\right). \tag{2.5.91}$$

The reason why we introduced this operator is that it can replace the operator $\mathcal{T}_0$ in (2.5.88) and (2.5.89) which solves the problem of piece-wise definition of $\mathcal{T}_0$. Indeed, firstly, thanks to Proposition 2.5.12, the similitude of the definition of $\mathcal{T}_0(x_\Gamma)$ given for $x_\Gamma \in \Gamma_{\mathrm{M}}$ by (2.5.48)-(2.5.50) and the one of the operator $\tilde{\mathcal{T}}_0$ given for $x_\Gamma \in \Gamma$ by (2.5.91), we directly can extend the proof of Proposition 2.5.3 for $\tilde{\mathcal{T}}_0$ and $x_\Gamma \in \Gamma$. Thus for all x_Γ, the operator $\tilde{\mathcal{T}}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma : \mathbb{H}\big(\hat{Y}_\infty\big) \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ is invertible and we have:

$$\sup_{x_\Gamma\in\Gamma}\left\|\big(\tilde{\mathcal{T}}_0(x_\Gamma) - \delta_\Sigma\otimes\delta_\Sigma\big)^{-1}\right\|_{\mathcal{L}\left(\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger,\mathbb{H}\left(\hat{Y}_\infty\right)\right)} < \infty. \tag{2.5.92}$$

Therefore it make a sense to write $\big(\tilde{\mathcal{T}}_0(x_\Gamma) - \delta_\Sigma \otimes \delta_\Sigma\big)^{-1}$.

Secondly, from the expression of these two operators given by (2.5.91) and (2.5.49), we have for all v independent of $\hat{\nu}$:

$$\forall x_\Gamma \in \Gamma \setminus \Gamma_{\mathrm{M}},\ \mathcal{T}_0(x_\Gamma)v = \tilde{\mathcal{T}}_0(x_\Gamma)v,$$

and by applying Proposition 2.5.12, we have for all v:

$$\forall x_\Gamma \in \Gamma_{\mathrm{M}},\ \mathcal{T}_0(x_\Gamma)v = \tilde{\mathcal{T}}_0(x_\Gamma)v.$$

Thus (2.5.88) and (2.5.89) can be rewritten as follow:

$$u_1(x_\Gamma;\cdot) := \big(\tilde{\mathcal{T}}_0(x_\Gamma) - \delta_\Sigma\otimes\delta_\Sigma\big)^{-1} y(x_\Gamma;\cdot) + \langle f_1(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty}(\hat{\nu}_+ - \hat{\nu}),$$

where we defined $y : \Gamma \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ for $x_\Gamma \in \Gamma$ by:

$$y(x_\Gamma;\cdot) := f_1(x_\Gamma;\cdot) - \langle f_1(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty}\delta_\Sigma + \tilde{\mathcal{T}}_0(x_\Gamma)d_{\hat{\nu}}^{-2}f_2(x_\Gamma;\cdot) - f_2(x_\Gamma;\cdot).$$

Thanks to this last expression and Proposition 2.5.13 it remains to prove:

$$x_\Gamma \in \Gamma \mapsto \big(\tilde{\mathcal{T}}_0(x_\Gamma) - \delta_\Sigma\otimes\delta_\Sigma\big)^{-1} \in C^m\Big(\mathcal{L}\big(\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger,\mathbb{H}\big(\hat{Y}_\infty\big)\big)\Big), \tag{2.5.93}$$

$$y \in H^m_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right), \tag{2.5.94}$$

$$x_\Gamma \in \Gamma \mapsto \langle f_1(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty}(\hat{\nu}_+ - \hat{\nu}) \in H^m_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)\right). \tag{2.5.95}$$

Indeed, thanks Proposition 2.5.12 and (2.5.91) we have that the map $x_\Gamma \in \Gamma \mapsto \tilde{\mathcal{T}}_0(x_\Gamma)$ belongs to $C^m\Big(\mathcal{L}\big(\mathbb{H}\big(\hat{Y}_\infty\big),\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\big)\Big)$. Therefore thanks to (2.5.92), re-solvent identity and the Leibniz formula, we conclude the proof of (2.5.93).

From the assumption $f_1 \in H^m_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right)$, we directly have that:

$$x_\Gamma \mapsto f_1(x_\Gamma;\cdot) - \langle f_1(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty}\delta_\Sigma \in H^m_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right). \tag{2.5.96}$$

Thanks to Proposition 2.5.12 and the similitude of the definition of $\mathcal{T}_0(x_\Gamma)$ given for $x_\Gamma \in \Gamma_M$ by (2.5.48)-(2.5.50) and the one of the operator $\tilde{\mathcal{T}}_0(x_\Gamma)$ given for $x_\Gamma \in \Gamma$ by (2.5.91), we directly can extend the proof of Proposition 2.5.7 for $\tilde{\mathcal{T}}_0$ and $x_\Gamma \in \Gamma$. Then from (2.5.70), the map $R : \Gamma \mapsto \mathcal{L}\left(\mathbb{C}_n[\hat{\nu}], \mathbb{H}(\hat{Y}_\infty)\right)$ defined for x_Γ by the linear operator $R(x_\Gamma)$ defined for $p \in \mathbb{C}_n[\hat{\nu}]$ by:

$$R(x_\Gamma)p := \tilde{\mathcal{T}}_0(x_\Gamma)d_{\hat{\nu}}^{-2}p - p,$$

belongs to $C^m\left(\Gamma; \mathcal{L}\left(\mathbb{C}_n[\hat{\nu}], \mathbb{H}(\hat{Y}_\infty)^\dagger\right)\right)$. Thus by applying Proposition 2.5.13 and using (2.5.96) we conclude the proof (2.5.94).

From the assumption $f_1 \in H^m_{0,\Gamma_M}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger\right)$ and $\hat{\nu} - \hat{\nu}_+ \in \mathbb{H}(\hat{Y}_\infty)$, we directly conclude the proof of (2.5.95). Thus whole the proof is finished. □

2.5.1.4 Existence of the near field

According to Proposition 2.5.11, to be able to assert that problem (2.3.31) is well posed, we have to show that:

$$\forall x_\Gamma \in \Gamma,\ r_k(x_\Gamma; \cdot) \in \mathbb{H}(\hat{Y}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}] \text{ where } r_k := \sum_{l=1}^{k} \mathcal{T}_l \hat{u}_{k-l}. \tag{2.5.97}$$

However to do so, we have to face a technical problem, namely the fact that $L^2(\hat{Y}_\infty)$ are not stable by multiplication with $\mathbb{C}[\hat{\nu}]$. For instance, the operator $\mathcal{T}_1$ is given for $\hat{v} \in H^1(\Gamma; \mathbb{H}(\hat{Y}_\infty))$ by:

$$\mathcal{T}_1\hat{v} := \widehat{\operatorname{div}}\left(\hat{\nu}\hat{\rho}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{v}\right) + \operatorname{div}_\Gamma\left(\hat{\rho}\,\widehat{\nabla}\,\hat{v}\right) + \widehat{\operatorname{div}}\left(\hat{\rho}\,\nabla_\Gamma\,\hat{v}\right),$$

and r_2 is given by: $r_2 = \mathcal{T}_1\hat{u}_1 + \mathcal{T}_2\hat{u}_0$. We cannot easily see that $\hat{\nu}\hat{\rho}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{u}_1 \in \left(L^2(\hat{Y}_\infty)\right)^3$. Thus we can not yet conclude that for all $x_\Gamma \in \Gamma$, $\widehat{\operatorname{div}}\left(\hat{\nu}\hat{\rho}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{u}_1\right)(x_\Gamma; \cdot) \in \mathbb{H}(\hat{Y}_\infty)^\dagger$ and so we also can not prove (2.5.97).

Nevertheless, this difficulty can be overcome by the following procedure: The property (2.5.97) will be proven by induction on k : this is the object of the forthcoming proposition Proposition 2.5.17, which will itself be a consequence of the next two propositions Proposition 2.5.15 and Proposition 2.5.16.

To state these propositions, we need to introduce a new notation. More precisely.

Definition 2.5.14. *Let $m \leq m_\Gamma$ and $u : \Gamma \mapsto \mathbb{H}_{\text{comp}}(\hat{Y}_\infty)^\dagger$. We say that u satisfies the $\mathcal{P}^\infty_m$ property if there exists $d \in \mathbb{N}$, a sequence $(u_l)_{l \in \mathbb{Z}^2\setminus\{0\}} \in H^m_{0,\Gamma_M}(\Gamma; \mathbb{C}_d[\hat{\nu}])$ such that:*

$$\forall (x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_+,\ u(x_\Gamma; \hat{x}, \hat{\nu}) = \sum_{l \in \mathbb{Z}^2\setminus\{0\}} u_l(x_\Gamma; \hat{\nu})\phi_l(x_\Gamma; \hat{x}, \hat{\nu}), \tag{2.5.98}$$

where we defined the sequence of functions $(\phi_l)_{l \in \mathbb{Z}^2\setminus\{0\}}$ for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_+$ by:

$$\phi_l(x_\Gamma, \hat{x}, \hat{\nu}) := e^{i2\pi l\hat{x}}e^{-2\pi\lambda_l(x_\Gamma)\hat{\nu}} \quad \textit{with} \quad \lambda_l(x_\Gamma) := |\operatorname{D}\psi_\Gamma(x_\Gamma)l|. \tag{2.5.99}$$

Moreover, the sequence of polynomial are required to satisfies:

$$\sum_{l\in\mathbb{Z}^2\setminus\{0\}} |l|^q \|u_l\|_{H^m(\Gamma;\mathbb{C}_d[\hat{\nu}])} < \infty. \tag{2.5.100}$$

In this last definition if u is not a function then (2.5.98) means: For all $\psi \in \mathcal{D}([0,1]^2\times]0,\infty[)\cap \mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$:

$$\langle u(x_\Gamma;\cdot),\psi\rangle_{\mathbb{H}_{\mathrm{comp}}\left(\hat{Y}_\infty\right)^\dagger-\mathbb{H}_{\mathrm{comp}}\left(\hat{Y}_\infty\right)} = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} \int_{\hat{Y}_\infty} u_l(x_\Gamma;\hat{\nu})\phi_l(x_\Gamma;\hat{x},\hat{\nu})\overline{\psi(\hat{x},\hat{\nu})}d\hat{x}d\hat{\nu}. \tag{2.5.101}$$

Then we have:

Proposition 2.5.15. *Let $1\le m\le m_\Gamma$, $d\in\mathbb{N}^*$ and assume that $u=u_1+u_2$ for some:*

$$(u_1,u_2)\in H^m_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)\Big)\times H^m\Big(\Gamma;\mathbb{C}_d[\hat{\nu}]\Big), \tag{2.5.102}$$

and u_1 satisfies $\mathcal{P}^\infty_m$ property. For all $k\in\mathbb{N}^*$ the following decomposition of $f^k := \mathcal{T}_k u$ holds:

$$f^k = f_1^k + f_2^k,$$

where:

- $f_2^k := \partial_{\hat{\nu}}\left(c^{(k)}\hat{\nu}^k\partial_{\hat{\nu}}u_2\right) + \mathrm{div}_\Gamma\left(\mathcal{C}^{(k-2)}\hat{\nu}^{(k-2)}\nabla_\Gamma u_2\right) + k^2c^{(k-2)}\hat{\nu}^{(k-2)}u_2.$
- $f_1^k \in H^q_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big)$ *satisfies the $\mathcal{P}^\infty_q$ property where we defined* $q := m-\min(2,k)$.
- *If $k+d\ge 2$ then*
$$f_2^k \in H^q\Big(\Gamma;\mathbb{C}_{k+d-2}[\hat{\nu}]\Big). \tag{2.5.103}$$
- *If $u_2=0$ or $d=0$ then:*
$$f_2^1 = 0. \tag{2.5.104}$$

The second useful proposition is the following one:

Proposition 2.5.16. *If the quantity f_1 appearing in Proposition 2.5.11 satisfies the $\mathcal{P}^\infty_m$ property then the quantity u_1 appearing in Proposition 2.5.11 also satisfies the $\mathcal{P}^\infty_m$ property.*

Indeed, thanks to Proposition 2.5.15 and Proposition 2.5.16 we can prove (2.5.97). Moreover, thanks to these last propositions we can easily prove the following result.

Proposition 2.5.17. *If the traces of far fields have the following regularity:*

$$\forall 0\le i\le n,\ x_\Gamma\in\Gamma\mapsto u_i(x_\Gamma,0)\in H^{m_\Gamma+\frac{1}{2}-i}(\Gamma), \tag{2.5.105}$$

then:

- *The sequence of near fields $(\hat{u}_i)_{0\le i\le n}$ can be defined for $x_\Gamma\in\Gamma$ with the following induction:*

$$\begin{cases} \hat{u}_0(x_\Gamma;\hat{x},\hat{\nu}) := u_0(x_\Gamma,0), \\ \forall 1\le k\le m_\Gamma,\ \hat{u}_k(x_\Gamma;\hat{x},\hat{\nu}) := u_k(x_\Gamma,0) - \displaystyle\sum_{i=1}^{k}\left(\mathcal{T}_0^{-1}\mathcal{T}_i\hat{u}_{k-i}\right)(x_\Gamma;\hat{x},\hat{\nu}). \end{cases} \tag{2.5.106}$$

- *There exists for all* $0 \le k \le m_\Gamma$*:*

$$(\hat{u}_k^1, p^k) \in H_{0,\Gamma_M}^{m_\Gamma+\frac{1}{2}-k}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\right) \times H^{m_\Gamma+\frac{1}{2}-k}\Big(\Gamma; \mathbb{C}_k[\hat{\nu}]\Big) \text{ such that } \hat{u}_k = \hat{u}_k^1 + p^k \tag{2.5.107}$$

- $\hat{u}_k^1$ *satisfies the* $\mathcal{P}^\infty_{m_\Gamma+\frac{1}{2}-k}$ *property.*

- *The required equation* (2.3.31) *is satisfied for all* $0 \le k \le m_\Gamma$*.*

Proof. We prove the result by induction on i. Let $0 \le i \le m_\Gamma$. The result is true for $i = 0$ because from (2.5.106) we have for all $x_\Gamma \in \Gamma$ $\hat{u}_0(x_\Gamma; \cdot) = u_0(x_\Gamma, 0)$ and from (2.5.105) this last quantity belongs to $\hat{u}_0 \in H^{m_\Gamma+\frac{1}{2}}(\Gamma)$.

Assume now for all $0 \le k \le i-1$, that (2.5.107) holds and $\hat{u}_k^1$ satisfies the $\mathcal{P}^\infty_{m_\Gamma+\frac{1}{2}-k}$ property. Therefore, applying Proposition 2.5.15, yields for all $1 \le k \le i$ the existence of:

$$\big(r_{k,i}^1, r_{k,i}^2\big) \in H_{0,\Gamma_M}^{q_{i,k}}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right) \times H^{q_{i,k}}\Big(\Gamma; \mathbb{C}_{\max(i-2,0)}[\hat{\nu}]\Big),$$

such that

$$\mathcal{T}_k \hat{u}_{i-k} = r_{k,l}^1 + r_{k,l}^2 : \tag{2.5.108}$$

with $q_{i,k} := m_\Gamma + \frac{1}{2} + k - i - \min(k, 2)$, $r_{k,l}^1$ satisfies the $\mathcal{P}^\infty_{q_{i,k}}$ property and for $i = 1$:

$$r_{k,1}^2 = 0. \tag{2.5.109}$$

Moreover we recall that:

$$r_i = \sum_{k=1}^{i} \mathcal{T}_i \hat{u}_{i-k},$$

and we emphasize that for all $1 \le k \le i$ that $q_{i,k} \ge m_\Gamma + \frac{1}{2} - i$. Thus thanks to (2.5.108) we have existence of:

$$\big(r_i^1, r_i^2\big) \in H_{0,\Gamma_M}^{m_\Gamma+\frac{1}{2}-i}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right) \times H^{m_\Gamma+\frac{1}{2}-i}\Big(\Gamma; \mathbb{C}_{\max(i-2,0)}[\hat{\nu}]\Big) \text{ such that } r_i = r_i^1 + r_i^2,$$

and thanks to (2.5.109) we have for $i = 1$, $r_1^2 = 0$. Therefore according to Proposition 2.5.16 we can define $\hat{v}^i := \mathcal{T}_0^{-1} r_i$. Thus we can define $\hat{u}_i$ by replacing $k = i$ in (2.5.106). Moreover there exist:

$$\big(\hat{u}_i^1, p^i\big) \in H_{0,\Gamma_M}^{m_\Gamma+\frac{1}{2}-i}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right) \times H^{m_\Gamma+\frac{1}{2}-i}\Big(\Gamma; \mathbb{C}_i[\hat{\nu}]\Big) \text{ such that } \hat{v}_i := \hat{u}_i^1 + p^i,$$

and $\hat{u}_i^1$ satisfies the $\mathcal{P}^\infty_{m_\Gamma+\frac{1}{2}-i}$ property.

Using the hypothesis (2.5.105) yields that the map: $p^i : \Gamma \mapsto \mathbb{C}_i[\hat{\nu}]$ defined for x_Γ by:

$$p^i(x_\Gamma; \cdot) := u_i(x_\Gamma, 0) + \hat{v}_i^2(x_\Gamma; \cdot),$$

belongs to $H^{m_\Gamma+\frac{1}{2}-i}\Big(\Gamma; \mathbb{C}_i[\hat{\nu}]\Big)$. We conclude our induction by remarking that $k = i$ in (2.5.106) implies $\hat{u}_i := \hat{u}_i^1 + p^i$.

Since (2.3.31) is a direct consequence of Proposition 2.5.10 then the proof is finished.□

A third useful result is the following one. It will be used in the proof of Proposition 2.5.15 and, later, for the derivation of our error estimates (cf. Chapter 3).

Proposition 2.5.18. *Let* $\hat{u} : \Gamma \mapsto \mathbb{H}_{\text{comp}}\big(\hat{Y}_\infty\big)^\dagger$ *and* $m \le m_\Gamma$*. If* $\hat{u}$ *satisfies the* $\mathcal{P}^\infty_m$ *property then for all* $\epsilon > 0$ *and* $r \in \mathbb{N}$ *we have :*

$$\forall r \in \mathbb{N},\ \exp\left(\pi g_{\min} \hat{\nu}\right) \hat{u} \in H^m\Big(\Gamma; C_b^r([0,1]^2 \times [\epsilon, \infty[)\Big).$$

2.5.1.5 Proof of Proposition 2.5.18

Let $\hat{u}$ satisfying the $\mathcal{P}_m^\infty$ property. Thanks to the Sobolev embedding theorem, it is sufficient to show that:

$$\forall r \in \mathbb{N},\ \exp\left(\pi g_{\min}\hat{\nu}\right)\hat{u} \in H^m\big(\Gamma; H^r(]0,1[^2\times[\epsilon,\infty[)\big). \tag{2.5.110}$$

To prove (2.5.110), we will use Proposition 2.5.13. Indeed, we can rewrite (2.5.98) as follow:

$$\forall x_\Gamma \in \Gamma,\ \hat{u}(x_\Gamma;\cdot) = A(x_\Gamma)u(x_\Gamma), \tag{2.5.111}$$

where $u = (u_l)_{l\in\mathbb{Z}^2\backslash\{0\}}$ is the sequence appearing in (2.5.98) and the map $x_\Gamma \mapsto A(x_\Gamma)$ is defined for $x_\Gamma \in \Gamma$ and sequence of polynomial $p = (p_l)_{l\in\mathbb{Z}^2\backslash\{0\}}$ by:

$$\forall\big(\hat{x},\hat{\nu}\big) \in \hat{Y}_+,\ \big(A(x_\Gamma)p\big)(\hat{x},\hat{\nu}) := \sum_{l\in\mathbb{Z}^2\backslash\{0\}} p_l(\hat{\nu})\phi_l(x_\Gamma;\hat{x},\hat{\nu}). \tag{2.5.112}$$

In order to have the convergence of the sum appearing in (2.5.112), the sequence of polynomial p is required to belongs to the following space:

$$E := \left\{ p \in \mathbb{C}_d[\hat{\nu}]^{\mathbb{Z}^2\backslash\{0\}},\ \|p\|_E^2 := \sum_{l\in\mathbb{Z}^2\backslash\{0\}} |l|^\alpha \|p_l\|^2_{\mathbb{C}_d[\hat{\nu}]} < \infty \right\},$$

where α and d are defined in the definition of the $\mathcal{P}_m^\infty$ property. Moreover, (2.5.110) is equivalent to:

$$\hat{u} \in C^{m_\Gamma}(\Gamma;F) \quad \text{with} \quad F := \exp\left(-g_{\min}\pi\epsilon\hat{\nu}\right)\cdot H^r(]0,1[^2\times]\epsilon,\infty[), \tag{2.5.113}$$

and from (2.5.100) we have

$$u \in H^m(\Gamma;E). \tag{2.5.114}$$

According to Proposition 2.5.13, (2.5.111) and (2.5.114), a sufficient condition to prove (2.5.113) is:

$$A \in C^{m_\Gamma}(\Gamma;\mathcal{L}(E,F)). \tag{2.5.115}$$

Therefore if we succeed to prove (2.5.115) then we will directly get (2.5.110). Hence, this will conclude our proof.

To prove (2.5.115), we apply now the following result:

Lemma 2.5.19. *Let F be a Hilbert space and $\alpha = (\alpha_l)_{l\in\mathbb{N}}$ such that:*

- *For all $l \in \mathbb{N}$, $\alpha_l \in C^{m_\Gamma}\big(\Gamma;F\big)$ and for all $(x_\Gamma,x'_\Gamma) \in \Gamma^2$:*

$$\forall(l,l') \in \mathbb{N}^2,\ l \neq l' \Rightarrow \big(\alpha_l(x_\Gamma),\alpha_{l'}(x'_\Gamma)\big)_F = 0. \tag{2.5.116}$$

- *There exists $q \in \mathbb{N}$ such that :*

$$\sum_{l\in\mathbb{N}} l^q \|\alpha_l\|^2_{C^{m_\Gamma}(\Gamma;F)} < \infty. \tag{2.5.117}$$

Define the space:

$$E_q := \left\{ p \in \mathbb{C}^{\mathbb{N}}, \|p\|_{E_q}^2 := \sum_{l\in\mathbb{N}} l^{-q}|p_l|^2 < \infty \right\}.$$

Then the map $x_\Gamma \mapsto T_\alpha(x_\Gamma)$ defined for x_Γ and $p \in E_q$ by:

$$T_\alpha(x_\Gamma)p := \sum_{l\in\mathbb{N}} p_l \alpha_l(x_\Gamma), \tag{2.5.118}$$

belongs to $C^{m_\Gamma}\big(\Gamma; \mathcal{L}(E_q, F)\big)$.

Proof. First, we emphasize that it is classical that E_q is a complete space. Then from the assumption of that F is complete, we deduce that $\mathcal{L}\big(E_q, F\big)$ is a Banach space. Therefore $C^{m_\Gamma}\big(\Gamma; \mathcal{L}(E_q, F)\big)$ is a Banach space.

Thus it remains to prove that the sequence $(T_\alpha^N)_{N\in\mathbb{N}}$ defined for $x_\Gamma \in \Gamma, p \in E_\alpha$ and $N \in \mathbb{N}$ by:

$$T_\alpha^N(x_\Gamma)p := \sum_{l=0}^{N} p_l \alpha_l(x_\Gamma), \tag{2.5.119}$$

is a Cauchy sequence in $C^{m_\Gamma}\big(\Gamma; \mathcal{L}(E_q, F)$. We recall that it mean:

$$\lim_{n\to\infty} \sup_{m\geq 0} \left\| T_\alpha^{m+n} - T_\alpha^n \right\|_{C^{m_\Gamma}\left(\Gamma;\mathcal{L}(E_q,F)\right)} = 0. \tag{2.5.120}$$

For all $l \in \mathbb{N}$ we assumed that $\alpha_l \in \mathbb{C}(\Gamma; F)$ and the sum appearing in (2.5.119) has a finite number of terms. Therefore for all $N \in \mathbb{N}$, we have $T_\alpha^N \in C^{m_\Gamma}\left(\Gamma; \mathcal{L}(E_\alpha, F)\right)$ and

$$\forall 0 \leq \gamma \leq m_\Gamma,\ \forall (p, x_\Gamma) \in E_q \times \Gamma,\ \big(\mathrm{D}_\gamma T_\alpha^N\big)(x_\Gamma)p := \sum_{l=0}^{N} p_l \mathrm{D}_\gamma \alpha_l(x_\Gamma).$$

Combining this with (2.5.116) yields for all $(m, n, x_\Gamma, p) \in \mathbb{N}^2 \times \Gamma \times E_q$ and $0 \leq \gamma \leq m_\Gamma$:

$$\begin{aligned}\left\| \big(\mathrm{D}_\gamma T_\alpha^{m+n}\big)(x_\Gamma)p - \big(\mathrm{D}_\gamma T_\alpha^n\big)(x_\Gamma)p \right\|_F^2 &= \sum_{l=n+1}^{n+m} \|p_l \mathrm{D}_\gamma \alpha_l(x_\Gamma)\|_F^2, \\ &\leq \sum_{l\in\mathbb{N}} |p_l|^2 l^{-q} \sum_{l=n+1}^{n+m} l^q \|\mathrm{D}_\gamma \alpha_l(x_\Gamma)\|_F^2.\end{aligned}$$

Thanks to the definition of the norm of the space E_q this yields for all $(m, n) \in \mathbb{N}^2$ and $0 \leq \gamma \leq m_\Gamma$:

$$\forall (p, x_\Gamma) \in E_q \times \Gamma,\ \left\| \big(\mathrm{D}_\gamma T_\alpha^{m+n}\big)(x_\Gamma)p - \big(\mathrm{D}_\gamma T_\alpha^n\big)(x_\Gamma)p \right\|_F^2 \leq \|p\|_{E_q} \sum_{l=n+1}^{n+m} l^q \|\mathrm{D}_\gamma \alpha_l(x_\Gamma)\|_F^2.$$

Thus by using the definition of the norm of the space $\mathcal{L}(E_q, F)$, this becomes :

$$\forall (m, n) \in \mathbb{N}^2,\ \forall x_\Gamma \in \Gamma \left\| \big(\mathrm{D}_\gamma T_\alpha^{m+n}\big)(x_\Gamma) - \big(\mathrm{D}_\gamma T_\alpha^n\big)(x_\Gamma) \right\|_{\mathcal{L}(E_q,F)}^2 \leq \sum_{l=n+1}^{n+m} l^q \|\mathrm{D}_\gamma \alpha_l(x_\Gamma)\|_{C^0(\Gamma;F)}^2.$$

By using the definition of the norm of the space $C^{m_\Gamma}\big(\Gamma;\mathcal{L}(E_q,F)\big)$ this yields:

$$\forall (m,n)\in\mathbb{N}^2,\ \left\|T_\alpha^{m+n}-T_\alpha^n\right\|^2_{C^{m_\Gamma}\left(\Gamma;\mathcal{L}(E_q,F)\right)} \leq \sum_{l=n+1}^{n+m} l^q\|\alpha_l\|^2_{C^{m_\Gamma}\left(\Gamma;\mathcal{L}(E_q,F)\right)}. \tag{2.5.121}$$

Moreover thanks to (2.5.117), we have:

$$\lim_{n\to\infty}\sup_{m\geq 0}\sum_{l=n+1}^{n+m} l^q\|\alpha_l\|_{C^{m_\Gamma}\left(\Gamma;\mathcal{L}(E_q,F)\right)}=0.$$

Combining this with (2.5.121) conclude the proof of (2.5.120). □

Indeed, thanks to the definition (2.5.118), we can rewrite for all $x_\Gamma\in\Gamma$ the definition of $A(x_\Gamma)$ as follow:

$$\forall u\in E, A(x_\Gamma)u=\sum_{k=0}^{d} T_{\phi^k}(x_\Gamma)u^k. \tag{2.5.122}$$

Here $(u^k)_{0\leq k\leq d}=\big((u_l^k)_{l\in\mathbb{Z}^2\setminus\{0\}}\big)_{0\leq k\leq d}$ is the unique element of $(E_\alpha)^{d+1}$ such that:

$$\forall l\in\mathbb{Z}^2\setminus\{0\},\ u_l=\sum_{k=0}^{q} u_l^k\hat{\nu}^k,$$

and $(\phi^k)_{0\leq k\leq d}=\big((\phi_l^k)_{l\in\mathbb{Z}^2\setminus\{0\}}\big)_{0\leq k\leq d}$ is defined for $0\leq k\leq d$, $l\in\mathbb{Z}^2\setminus\{0\}$, $(x_\Gamma;\hat{x},\hat{\nu})\in\Gamma\times\hat{Y}_+$ by:

$$\phi_l^k(x_\Gamma;\hat{x},\hat{\nu}):=\hat{\nu}^k\phi_l(x_\Gamma;\hat{x},\hat{\nu}).$$

Thanks to (2.5.122) a sufficient condition to prove (2.5.115) is:

$$\forall 0\leq k\leq d,\ T_{\phi^k}\in C\big(\Gamma;\mathcal{L}(E_\alpha,F)\big). \tag{2.5.123}$$

Let $0\leq k\leq d$. According to Lemma 2.5.19 with $\alpha=\phi^q$, to prove (2.5.123), we now proceed as follow:

1. We prove for all $0\leq\gamma\leq m_\Gamma$ the existence of a multivariate polynomial $P_{\gamma,\nu}$ such that
$$\forall l\in\mathbb{Z}^2\setminus\{0\},\ \mathrm{D}^\gamma\phi_l^k=P_{\gamma,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat{\nu}\big)\phi_l^k. \tag{2.5.124}$$

2. We prove that for all $\gamma\in\mathbb{N}$ we have the existence of a multivariate polynomial P_γ such that for all $l\in\mathbb{Z}^2$ we have:
$$\hat{\mathrm{D}}^\gamma\big(\exp\left(\pi g_{\min}\hat{\nu}\right)\phi_l^k\big)=P_\gamma(\lambda_l,l,\hat{\nu})\phi_l\exp\left(\pi g_{\min}\hat{\nu}\right). \tag{2.5.125}$$

3. We deduce that for all $l\in\mathbb{Z}^2\setminus\{0\}$:
$$\phi_l^k\in C^{m_\Gamma}(\Gamma;F), \tag{2.5.126}$$
with the existence of $C>0$ independent of l such that :
$$\|\phi_l^k\|_{C^{m_\Gamma}(\Gamma;F)}\leq C\exp\left(-\frac{\pi}{2}\epsilon g_{\min}|l|\right). \tag{2.5.127}$$

4. From (2.5.127) we directly have that (2.5.117) holds:
$$\sum_{l\in\mathbb{Z}^2\setminus\{0\}} l^{\alpha}\|\phi_l^k\|_{C^{m_\Gamma}(\Gamma;F)} < \infty. \tag{2.5.128}$$

5. We prove that (2.5.116) holds: For all $(x_\Gamma, x'_\Gamma)\in\Gamma^2$: and:
$$\forall (l,l')\in \big(\mathbb{Z}^2\setminus\{0\}\big)^2,\ l\neq l' \Rightarrow \big(\phi_l^k(x_\Gamma;\cdot),\phi_{l'}^q(x'_\Gamma;\cdot)\big)_F = 0. \tag{2.5.129}$$

Proof of (2.5.124). We prove this result with an induction on γ. The result is trivial for $\gamma=0$. Let γ such that (2.5.124) holds. Let $x_\Gamma\in\Gamma$ and $v\in T_{x_\Gamma}\Gamma$. We define for any quantity f differentiable at the point x_Γ:
$$\partial_v f(x_\Gamma) := \mathrm{D}\, f(x_\Gamma)\cdot v. \tag{2.5.130}$$
Let $l\in\mathbb{Z}^2\setminus\{0\}$. It is classical that ∂_v satisfies the the Leibniz formula. Therefore (2.5.124) yields:
$$\begin{aligned}\partial_v\big(\mathrm{D}^\gamma\,\phi_l^k\big)(x_\Gamma;\cdot) &= \partial_v\Big(P_{\gamma,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)\Big)(x_\Gamma)\phi_l^k(x_\Gamma;\cdot),\\ &\quad + P_{\gamma,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)(x_\Gamma)\partial_v(\phi_l^k)(x_\Gamma;\cdot),\end{aligned} \tag{2.5.131}$$
According to the chain rule formula and (2.5.130), we have:
$$\begin{aligned}\partial_v\Big(P_{\gamma,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)\Big)(x_\Gamma) &:= \sum_{q=0}^{\gamma}\big(\mathrm{D}_{X_q}\,P_{\gamma,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)\big)\,(x_\Gamma)\cdot\partial_v\,\mathrm{D}^q\,\lambda_q(x_\Gamma),\\ &= P^1_{\gamma+1,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^{\gamma+1}\,\lambda_l,\hat\nu\big)(x_\Gamma)\cdot v,\end{aligned} \tag{2.5.132}$$
where we defined for $(X_0,\cdots,X_{\gamma+1})$ and $t\in T_{x_\Gamma}\Gamma$:
$$P^1_{\gamma+1,d}\big(X_0,\cdots,X_{\gamma+1},\hat\nu\big)\cdot t := \sum_{q=0}^{\gamma}\Big(\mathrm{D}_{X_q}\,P_{\gamma,d}\big(X_0,\cdots,X_\gamma,\hat\nu\big)\Big)\cdot(X_{q+1}\cdot t)$$
Moreover from the definition of ϕ_l^k, we have $\partial_v(\phi_l^k)(x_\Gamma;\cdot) = -2\pi\hat\nu\partial_v\lambda_l\phi_l^k(x_\Gamma;\cdot)$. Therefore:
$$P_{\gamma,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)\partial_v(\phi_l^k) = \big(P^2_{\gamma+1,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^{\gamma+1}\,\lambda_l,\hat\nu\big)\cdot v\big)\,\phi_l^k, \tag{2.5.133}$$
where we defined for $(X_0,\cdots,X_{\gamma+1})$ and $t\in T_{x_\Gamma}\Gamma$:
$$P^2_{\gamma+1,d}\big(X_0,\cdots,X_{\gamma+1},\hat\nu\big)\cdot t := -2\pi\hat\nu P_{\gamma,d}\big(X_0,\cdots,X_\gamma,\hat\nu\big)\cdot(X_1\cdot t)$$
Thus combining (2.5.131), (2.5.132) and (2.5.133) yields:
$$\forall v\in T_{x_\Gamma}\Gamma,\ \partial_v\big(\mathrm{D}^\gamma\,\phi_l^k\big)(x_\Gamma;\cdot) = P_{\gamma+1,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)(x_\Gamma)\phi_l^k(x_\Gamma;\cdot)\cdot v.$$
where we defined $P_{\gamma+1,d} := P^1_{\gamma+1,d}+P^2_{\gamma+1,d}$. Therefore according to the definition (2.5.130), we have:
$$\mathrm{D}^{\gamma+1}\,\big(\phi_l^k\big)(x_\Gamma;\cdot) = P_{\gamma+1,d}\big(\lambda_l,\mathrm{D}\,\lambda_l,\cdots,\mathrm{D}^\gamma\,\lambda_l,\hat\nu\big)\phi_l^k,$$
which conclude the proof of our induction.

Proof of (2.5.125). The proof of this result is exactly the same as the proof of (2.5.124). Therefore we do not present it here.

Proof of (2.5.126) **and** (2.5.127)

We need the following result:

Lemma 2.5.20. *There exists $C > 0$ such that for all $l \in \mathbb{Z}^2 \setminus \{0\}$ the following estimate holds:*

$$\|\lambda_l\|_{C^{m_\Gamma}(\Gamma_\mathrm{M})} \leq C|l|.$$

Proof. We recall that for all $l \in \mathbb{Z}^2 \setminus \{0\}$ the function λ_l is given by:

$$\lambda_l = \sqrt{(\mathrm{D}\,\psi_\Gamma l, \mathrm{D}\,\psi_\Gamma l)}.$$

We introduce the function $\lambda : S^1 \times \overline{\Gamma_\mathrm{M}}$ defined for $(\hat{l}, x_\Gamma) \in S^1 \times \overline{\Gamma_\mathrm{M}}$ by $|\mathrm{D}\,\psi_\Gamma \hat{l}|$ in order to have the following rewriting:

$$\forall l \in \mathbb{N},\ \lambda_l(x_\Gamma) = |l|\lambda\left(\frac{l}{|l|}, x_\Gamma\right). \tag{2.5.134}$$

Let us proof that $\lambda \in C^{m_\Gamma}(\Gamma)$. Indeed on one hand the regularity of function ψ_Γ implies that the function defined for $(\hat{l}, x_\Gamma) \in S^1 \times \overline{\Gamma_\mathrm{M}}$ by $(\mathrm{D}\,\psi_\Gamma \hat{l}, \mathrm{D}\,\psi_\Gamma \hat{l})$ belongs to $C^{m_\Gamma}(S^1 \times \Gamma)$. On the other hand we get thanks to the definition of the set Γ_M and the regularity of the function ψ_Γ existence of a constant $c > 0$ such that:

$$\forall (\hat{l}, x_\Gamma) \in S^1 \times \Gamma_\mathrm{M}\, c^{-1},\ \leq (\mathrm{D}\,\psi_\Gamma(x_\Gamma)\hat{l}, \mathrm{D}\,\psi_\Gamma(x_\Gamma)\hat{l}) \leq c.$$

Moreover the square-root is of class C^∞ on the interval $[c^{-1}, c]$ and the function lambda is the composition of this last function with the square root which conclude the proof of $\lambda \in C^{m_\Gamma}(\Gamma_\mathrm{M})$.

Combining this with the compactness of $\overline{\Gamma_\mathrm{M}} \times S^1$ yields :

$$\sup_{\hat{l} \in S^1} \|\lambda_{\hat{l}}\|_{C^{m_\Gamma}(\Gamma_\mathrm{M})} = \sup_{(\hat{l}, x_\Gamma) \in S^1 \times \overline{\Gamma_\mathrm{M}}, m' \leq m_\Gamma} |\mathrm{D}_{x_\Gamma}^{m'} \lambda_{\hat{l}}(x_\Gamma)| \leq \|\lambda\|_{C^{m_\Gamma}(S^1 \times \Gamma_\mathrm{M})}.$$

Thus combining this last estimate with (2.5.134) yields the following result:

$$\|\lambda_l\|_{C^{m_\Gamma}(\Gamma_\mathrm{M})} \leq \|\lambda\|_{C^{m_\Gamma}(S^1 \times \overline{\Gamma_\mathrm{M}})} |l|,$$

which is the stated result. $\square$

Thanks to (2.5.124) and (2.5.125), for all $0 \leq \gamma \leq m_\Gamma$ and $\gamma' \in \mathbb{N}$ we have the existence of $N \in \mathbb{N}$ and $C > 0$ such that for all $l \in \mathbb{Z}^2 \setminus \{0\}$ and $(x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_+$:

$$\left|\mathrm{D}^\gamma\, \hat{\mathrm{D}}^{\gamma'} \big(\exp(\pi g_{\min} \hat{\nu})\, \phi_l^k\big)(x_\Gamma; \hat{x}, \hat{\nu})\right|^2 \leq C\left(\hat{\nu}^{2N} + |l|^{2N}\right) \exp(-2\pi |l| g_{\min} \hat{\nu}). \tag{2.5.135}$$

Moreover, by using the following proposition:

$$\forall N \in \mathbb{N},\ \lim_{x \to \infty} x^N \exp(-x) = 0,$$

we can prove that

$$C_{N,\epsilon} := \sup_{(\hat{\nu}, l) \in [\epsilon, \infty[\times [1, \infty[} \left(\hat{\nu}^{2N} + |l|^{2N}\right) \exp\left(-\frac{\pi g_{\min}}{2} \hat{\nu} l\right) < \infty.$$

Thus there exists $C > 0$ such that for all $\hat{\nu} > \epsilon$ and $l \in \mathbb{Z}^2$:

$$\left(\hat{\nu}^{2N} + |l|^{2N}\right) \exp(-2\pi |l| g_{\min} \hat{\nu}) \leq C \exp(-\pi \epsilon g_{\min} |l|) \exp\left(-\frac{\pi g_{\min}}{2} \hat{\nu}\right).$$

Combining this with (2.5.135) yields the existence of $C > 0$ such that for all $l \in \mathbb{Z}^2 \setminus \{0\}$:

$$\int_{]0,1[^2 \times]\epsilon,\infty[} \sup_{x_\Gamma \in \Gamma} \left| \mathrm{D}^\gamma \hat{\mathrm{D}^{\gamma'}} \big(\exp(\pi g_{\min} \hat{\nu})\, \phi_l^k \big)(x_\Gamma; \hat{x}, \hat{\nu}) \right|^2 d\hat{x} d\hat{\nu} \leq C \exp(-\pi g_{\min} |l|). \quad (2.5.136)$$

According to the Dominated convergence theorem, this last estimate concludes the proof of (2.5.126). The estimate (2.5.127) is also a direct consequence of (2.5.136).

Proof of (2.5.129). This property is a direct consequence of (2.5.125) and orthogonality on $L^2(]0,1[^2)$ of the family $(\hat{x} \to \exp(2\pi i l \cdot \hat{x}), l \in \mathbb{Z}^2)$.

Since we succeeded to prove (2.5.128) and (2.5.129), we now can apply Lemma 2.5.19. Therefore we conclude the proof of (2.5.115) and we have seen that (2.5.115) is was a sufficient condition to prove (2.5.110). Therefore we conclude the proof of Proposition 2.5.18.

2.5.1.6 Proof of Proposition 2.5.16

Let $x_\Gamma \in \Gamma$. Since we have by construction $\hat{u}(x_\Gamma; \cdot) - \hat{u}_2(x_\Gamma; \hat{\nu}) \in \mathbb{H}(\hat{Y}_\infty)$ then we have $\hat{u}(x_\Gamma; \cdot) \in L^2_{\text{loc}}(\hat{\Omega})$. Therefore according to the Fubini theorem we have for all $\hat{\nu} > 0$ that the map $\hat{x} \in]0,1[^2 \mapsto \hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) \in L^2(]0,1[^2)$. Moreover, we recall $(\hat{x} \to \exp(2\pi i l \cdot \hat{x}), l \in \mathbb{Z}^2)$ is a Hilbert basis of $L^2(]0,1[^2)$. Therefore for all $(\hat{x}, \hat{\nu}) \in \hat{Y}_\infty$ we have:

$$\hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) = \sum_{l \in \mathbb{Z}^2 \setminus \{0\}} \exp\big(\lambda_l(x_\Gamma)\big) \hat{u}_l(x_\Gamma; \hat{\nu}) + \hat{u}_2(x_\Gamma; \hat{\nu}), \quad (2.5.137)$$

where we defined for $l \in \mathbb{N}, x_\Gamma \in \Gamma$ the following quantity :

$$\hat{u}_l(x_\Gamma; \hat{\nu}) := \exp\big(-\lambda_l(x_\Gamma)\hat{\nu}\big) \int_{]0,1[^2 \times \{\hat{\nu}\}} \exp(-i2\pi l \hat{x}) \hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) \mathrm{d}\hat{x}. \quad (2.5.138)$$

Therefore, we now prove that for all $l \in \mathbb{Z}^2 \setminus \{0\}$:

$$\partial^2_{\hat{\nu}} \hat{u}_l(x_\Gamma; \hat{\nu}) - 2\pi \lambda_l(x_\Gamma) \partial_{\hat{\nu}} \hat{u}_l(x_\Gamma; \hat{\nu}) = f_l(x_\Gamma; \hat{\nu}), \quad (2.5.139)$$

$$\hat{u}_l(x_\Gamma; \hat{\nu}) \in \mathbb{C}_d[\hat{\nu}], \quad (2.5.140)$$

$$x_\Gamma \mapsto u_l(x_\Gamma; \hat{\nu}) \in C^{m_\Gamma}\big(\Gamma; \mathbb{C}_d[\hat{\nu}]\big), \quad (2.5.141)$$

$$\|\hat{u}_l\|_{C^{m_\Gamma}\left(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]\right)} \leq C \left(1 + |l| \|f_l\|_{C^{m_\Gamma}\left(\Gamma; \mathbb{C}_d[\hat{\nu}]\right)}\right). \quad (2.5.142)$$

In (2.5.142), C is a constant independent of l. Finally, we will prove the existence of $q' \in \mathbb{Z}$ such that:

$$\sum_{l \in \mathbb{Z}^2 \setminus \{0\}} |l|^{q'} \|\hat{u}_l\|^2_{C^{m_\Gamma}\left(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]\right)} < \infty. \quad (2.5.143)$$

Proof of (2.5.139)**.** We emphasize that according to Proposition 2.5.18 and (2.5.101) that $f(x_\Gamma; \cdot) \in L^2_{\text{loc}}(\hat{\Omega})$ and this function is defined for $(\hat{x}, \hat{\nu}) \in \hat{Y}_\infty$ by:

$$f(x_\Gamma; \hat{x}, \hat{\nu}) = \sum_{l \in \mathbb{Z}^2 \setminus \{0\}} f_l(x_\Gamma; \hat{\nu}) \phi_l(x_\Gamma; \hat{x}, \hat{\nu}), \quad (2.5.144)$$

and for all $l \in \mathbb{Z}^2$ we have:

$$f_l(x_\Gamma, \hat{\nu}) := \exp\big(-\lambda_l(x_\Gamma)\hat{\nu}\big) \int_{]0,1[^2 \times \{\hat{\nu}\}} \exp(-i2\pi l \hat{x}) f(x_\Gamma; \hat{x}, \hat{\nu}) \mathrm{d}\hat{x}. \quad (2.5.145)$$

Therefore thanks to $\mathcal{T}_0(x_\Gamma)u(x_\Gamma;\cdot) = f(x_\Gamma;\cdot)$, (2.5.48) and (2.5.50) the following equality holds:

$$\forall(\hat{x},\hat{\nu}) \in \hat{\Omega},\ \partial^2_{\hat{\nu}}\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) + \sum_{ij}\partial_{\hat{x}_j}\big(G_{ij}\partial_{\hat{x}_i}\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\big) = f(x_\Gamma;\hat{x},\hat{\nu}), \tag{2.5.146}$$

where $G = \big(G_{ij}\big)_{ij}$ is given by:

$$G := \mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger. \tag{2.5.147}$$

Thanks to $f(x_\Gamma;\cdot) \in L^2_{\mathrm{loc}}(\hat{\Omega})$ and (2.5.146), the result of regularity for elliptic operators yields that $\hat{u}(x_\Gamma;\cdot) \in H^2_{\mathrm{loc}}(\hat{\Omega})$. Thus, using that $\hat{u}(x_\Gamma;\cdot)$ and $\hat{x} \mapsto \exp(2\pi i l\cdot\hat{x})$ are both one periodic on the variable $\hat{x}$ yields for all i,j:

$$\int_{]0,1[^2\times\{\hat{\nu}\}}\partial_{\hat{x}_j}\big(G_{ij}\partial_{\hat{x}_i}\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\big)\exp(-2\pi i l\cdot\hat{x})\mathrm{d}\hat{x} = \int_{]0,1[^2\times\{\hat{\nu}\}}\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\partial_{\hat{x}_j}\big(G_{ij}\partial_{\hat{x}_i}\exp(-2\pi i l\cdot\hat{x})\big)\mathrm{d}\hat{x}. \tag{2.5.148}$$

Moreover, thanks to (2.5.147) and the definition of λ_l given in (2.5.99), we have

$$\partial_{\hat{x}_j}\big(G_{ij}\partial_{\hat{x}_i}\exp(-2\pi i l\cdot\hat{x})\big) = -(2\pi)^2(Gl,l) = -\big(2\pi\lambda_l(x_\Gamma)\big)^2.$$

Combining this with (2.5.138) yields that (2.5.148) becomes:

$$\int_{]0,1[^2\times\{\hat{\nu}\}}\partial_{\hat{x}_j}\big(G_{ij}\partial_{\hat{x}_i}\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\big)\exp(-2\pi i l\cdot\hat{x})\mathrm{d}\hat{x} = -\big(2\pi\lambda_l(x_\Gamma)\big)^2\exp\big(\lambda_l(x_\Gamma)\hat{\nu}\big)\hat{u}_l(x_\Gamma;\hat{\nu}).$$

Combining this with (2.5.146), (2.5.138) and (2.5.145) leads to the following differential equation:

$$\partial^2_{\hat{\nu}}\left(\exp\big(\lambda_l(x_\Gamma)\hat{\nu}\big)\hat{u}_l(x_\Gamma;\hat{\nu})\right) - \big(2\pi\lambda_l(x_\Gamma)\big)^2\exp\big(\lambda_l(x_\Gamma)\hat{\nu}\big)\hat{u}_l(x_\Gamma;\hat{\nu}) = \exp\big(\lambda_l(x_\Gamma)\hat{\nu}\big)f_l(x_\Gamma;\hat{\nu}).$$

From this last equation we directly get (2.5.139).

Proof of (2.5.140).

We now build a particular solution $\hat{u}_l^{\mathrm{part}}(x_\Gamma;\hat{\nu})$ of (2.5.139) taking the form:

$$\hat{u}_l^{\mathrm{part}}(x_\Gamma;\hat{\nu}) = \hat{R}_l(x_\Gamma)f_l(x_\Gamma;\hat{\nu}),$$

for some $\hat{R}_l(x_\Gamma) \in \mathcal{L}\big(\mathbb{C}_d[\hat{\nu}],\mathbb{C}_{d+1}[\hat{\nu}]\big)$.

We denote for $d' \in \mathbb{N}$ by $\mathrm{D}_{d'}$ the derivative operator on the space $\mathbb{C}_{d'}[\hat{\nu}]$ in order to rewrite (2.5.139) as follow:

$$\big(\mathrm{D}_d - 2\pi\lambda_l(x_\Gamma)\big)\mathrm{D}_{d+1}\hat{u}_l(x_\Gamma;\hat{\nu}) = f_l(x_\Gamma;\hat{\nu}). \tag{2.5.149}$$

Since D_d is nilpotent of order $d+1$, the operator $\mathrm{D}_d - 2\pi\lambda_l(x_\Gamma) : \mathbb{C}_d[\hat{\nu}] \mapsto \mathbb{C}_d[\hat{\nu}]$ is invertible and its inverse is given by:

$$\big(\mathrm{D}_d - 2\pi\lambda_l(x_\Gamma)\big)^{-1} = -\sum_{p=0}^{d}\big(2\pi\lambda_l(x_\Gamma)\big)^{-(p+1)}\mathrm{D}_d^p. \tag{2.5.150}$$

We introduce the integrator operator $\mathrm{Int}_d \in \mathcal{L}(\mathbb{C}_d[\hat{\nu}], \mathbb{C}_{d+1}[\hat{\nu}])$ defined for polynomial $P \in \mathbb{C}_d[\hat{\nu}]$ by the unique solution of: Find Q such that

$$Q' = P \text{ with the initial condition } Q(0) = 0. \tag{2.5.151}$$

Thus we now can define $\hat{R}_l(x_\Gamma) \in \mathcal{L}(\mathbb{C}_d[\hat{\nu}], \mathbb{C}_{d+1}[\hat{\nu}])$ by:

$$\hat{R}_l(x_\Gamma) := -\mathrm{Int}_d \sum_{p=0}^{d} \big(2\pi\lambda_l(x_\Gamma)\big)^{-(p+1)} \mathrm{D}_n^p, \tag{2.5.152}$$

and thanks to (2.5.150) and (2.5.151) this last operator satisfies:

$$\big(\mathrm{D}_d - 2\pi\lambda_l(x_\Gamma)\big)\mathrm{D}_{d+1}\hat{u}_l(x_\Gamma;\hat{\nu})\hat{R}_l(x_\Gamma) = \mathbb{I}_{\mathbb{C}_d[\hat{\nu}]}.$$

Thus $\hat{u}^l_{\mathrm{part}}(x_\Gamma;\hat{\nu}) := \hat{R}_l(x_\Gamma) f_l(x_\Gamma;\hat{\nu})$ is a solution (2.5.149). Therefore $\hat{u}^l_{\mathrm{part}}(x_\Gamma;\hat{\nu})$ satisfies (2.5.139). Since $\hat{u}_l(x_\Gamma;\hat{\nu})$ and $\hat{u}^l_{\mathrm{part}}(x_\Gamma;\hat{\nu})$ are both solutions of 2.5.139 then there exists A and B such that: $\hat{u}_l = A + B\exp(2\pi\hat{\nu}\lambda_l) + \hat{u}_l^{\mathrm{part}}$. Since $\hat{u}(x_\Gamma;) \in \mathbb{H}\big(\hat{Y}_\infty\big)$ we directly have that $B = 0$. From the initial condition (2.5.151), we have $\hat{u}^l_{\mathrm{part}}(x_\Gamma;0) = 0$. Therefore:

$$\hat{u}_l(x_\Gamma;\hat{\nu}) = \hat{u}_l(x_\Gamma;0) + \hat{u}_l^{\mathrm{part}}(x_\Gamma;\hat{\nu}), \tag{2.5.153}$$

where we recall that:

$$\hat{u}_l(x_\Gamma;0) = \int_\Sigma \exp(-i2\pi l\hat{x})\hat{u}(x_\Gamma;\hat{x},0)d\hat{x}.$$

Proof of (2.5.141) **and** (2.5.142). Let us prove that:

$$\hat{R}_l := x_\Gamma \in \Gamma \mapsto \hat{R}_l(x_\Gamma) \in C^{m_\Gamma}\Big(\Gamma; \mathcal{L}\big(\mathbb{C}_d[\hat{\nu}], \mathbb{C}_{d+1}[\hat{\nu}]\big)\Big) \tag{2.5.154}$$

with the existence of $C > 0$ independent of l such that

$$\|\hat{R}_l\|_{C^{m_\Gamma}\Big(\Gamma;\mathcal{L}\big(\mathbb{C}_d[\hat{\nu}],\mathbb{C}_{d+1}[\hat{\nu}]\big)\Big)} \leq C|l|. \tag{2.5.155}$$

Indeed, (2.5.154) is a direct consequence of $\lambda_l \in C^{m_\Gamma}(\Gamma)$ and (2.5.152). Moreover we can prove by induction on m_Γ the existence of $C > 0$ independent of l such that:

$$\forall 0 \leq p \leq d,\ \|\lambda_l^{-(p+1)}\|_{C^{m_\Gamma}(\Gamma)} \leq C\|\lambda_l\|_{C^{m_\Gamma}(\Gamma)}.$$

Combining this with Proposition 2.5.20 and (2.5.152) conclude the proof of (2.5.155).

Now let us prove that:

$$\hat{u}_l^0 := x_\Gamma \mapsto \hat{u}_l(x_\Gamma;0) \in C^{m_\Gamma}(\Gamma), \tag{2.5.156}$$

with the existence of $C > 0$ independent of l such that:

$$\|\hat{u}_l^0\|_{C^{m_\Gamma}(\Gamma)} \leq C. \tag{2.5.157}$$

Indeed, we have:

$$\forall x_\Gamma \in \Gamma,\ \hat{u}_l^0(x_\Gamma) := \langle L_l, \hat{u}(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty}$$

where $L_l \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ is defined for $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\langle L_l, v\rangle_{\hat{Y}_\infty} := \int_\Sigma \exp(-i2\pi l\hat{x})v(\hat{x},0)d\hat{x}.$$

Thanks to the continuity of the trace operator $\mathbb{H}\big(\hat{Y}_\infty\big) \mapsto L^2(\Gamma)$, we have existence of $C > 0$ independent of l such that:

$$\|L_l\|_{\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger} \leq C.$$

According to Proposition 2.5.13, combining this with $\hat{u} \in C^{m_\Gamma}(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big))$ conclude the proof (2.5.156) and (2.5.157).

According to Proposition 2.5.13, combining (2.5.154)-(2.5.155) with $f_l \in C^{m_\Gamma}(\Gamma; \mathbb{C}_d[\hat{\nu}])$ yields:

$$\hat{u}^l_{\text{part}} \in C^{m_\Gamma}(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]), \tag{2.5.158}$$

with the existence of $C > 0$ independent of l such that:

$$\|\hat{u}^l_{\text{part}}\|_{C^{m_\Gamma}(\Gamma;\mathbb{C}_{d+1}[\hat{\nu}])} \leq C|l|\|f_l\|_{C^{m_\Gamma}(\Gamma;\mathbb{C}_d[\hat{\nu}])}. \tag{2.5.159}$$

Combining (2.5.158) and (2.5.156) with (2.5.153) conclude the proof of (2.5.141). Combining (2.5.157) and (2.5.159) with (2.5.153) conclude the proof of (2.5.142).

Proof of (2.5.143). This a direct consequence of (2.5.142) and the assumption:

$$\exists q \in \mathbb{Z}, \sum_{l\in\mathbb{Z}^2\setminus\{0\}} |l|^q \|f_l\|^2_{C^{m_\Gamma}\left(\Gamma;\mathbb{C}_d[\hat{\nu}]\right)} < \infty.$$

2.5.1.7 Proof of Proposition 2.5.15

The property (2.5.103) and (2.5.104) are direct consequence of the expression of the quantity f^k_2. For the rest of the proof Proposition 2.5.15 we proceed as follow:

1. We prove for all $\epsilon > 0$, that:

$$\chi^\epsilon f^k_1 \in H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right), \tag{2.5.160}$$

   where $(\chi^\epsilon)_\epsilon$ is a sequence of C^∞ cut off function such that for all $\epsilon > 0$ we have $\chi^\epsilon \equiv 1$ on $[-1, \epsilon]$ and $\chi^\epsilon \equiv 0$ on $[2 \cdot \epsilon, \infty[$.

2. We prove that f^k_1 satisfies the $\mathcal{P}^\infty_q$ property.

3. We prove :

$$(1 - \chi^\epsilon)f^k_1 \in H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right). \tag{2.5.161}$$

4. We deduce that $f^k_1 \in H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right)$ from (2.5.160) and (2.5.161).

Proof of (2.5.160). Let $\epsilon > 0$. Using that $\mathrm{supp}(\chi^\epsilon) = [-1, 2\cdot\epsilon]$ yields that this last quantities is equal to:

$$\chi^\epsilon\Big(\mathcal{T}_{k,\#}(\chi^{2\epsilon}u) - f_2^k\Big) \quad \text{with} \quad \mathcal{T}_{k,\#} := \sum_{i=0}^{3} \mathcal{T}_{k,\#}^i.$$

Here we defined for $\hat{v}$ smooth enough:

$$\begin{cases} \mathcal{T}_{k,\#}^0 \hat{v} := \widehat{\mathrm{div}}\left(I_\epsilon \hat{\rho}\mathcal{C}^{(k)}\hat{\nu}^k\, \widehat{\nabla}\, \hat{v}\right), \\ \mathcal{T}_{k,\#}^1 \hat{v} := \mathrm{div}_\Gamma\left(I_\epsilon \hat{\rho}\mathcal{C}^{(k-1)}\hat{\nu}^{k-1}\, \widehat{\nabla}\, \hat{v}\right), \\ \mathcal{T}_{k,\#}^2 \hat{v} := \widehat{\mathrm{div}}\left(I_\epsilon \hat{\rho}\mathcal{C}^{(k-1)}\hat{\nu}^{k-1}\, \nabla_\Gamma\, \hat{v}\right), \\ \mathcal{T}_{k,\#}^3 \hat{v} := \mathrm{div}_\Gamma\left(I_\epsilon \hat{\rho}\mathcal{C}^{(k-2)}\hat{\nu}^{k-2}\, \nabla_\Gamma\, \hat{v}\right) + k^2 I_\epsilon \hat{\mu} c^{((p-2))}\hat{\nu}^{k-2}\hat{v}, \end{cases}$$

and I_ϵ is the indicator function of $[-1, 2\cdot\epsilon]$. From the assumption (2.5.102) and the regularity of the function $\chi^{2\epsilon}$ we can easily prove that:

$$\chi^{2\epsilon}u \in H^m_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)\right). \tag{2.5.162}$$

For all p we recall that the quantity $\mathcal{C}^{(p)}$ and $c^{(p)}$ both belong to $C^{m_\Gamma}\Big(\Gamma; L^\infty(\hat{Y}_\infty)\Big)$. Therefore for the quantity $I_\epsilon\hat{\rho}\mathcal{C}^{(p)}\hat{\nu}^p$ and $I_\epsilon\hat{\mu}c^{(p)}\hat{\nu}^p\hat{v}$ both belong to $C^{m_\Gamma}\Big(\Gamma; L^\infty(\hat{Y}_\infty)\Big)$. Therefore, by using Proposition 2.5.13, we can prove that:

$$\forall 0 \le p \le 2, \mathcal{T}_{k,\#}^p \in \mathcal{L}\Big(H^m_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)\right), H^{m-1}_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger\right)\Big),$$

and

$$\mathcal{T}_{k,\#}^3 \in \mathcal{L}\Big(H^m_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)\right), H^{m-2}_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger\right)\Big).$$

Moreover we recall that $\mathcal{T}_{1,\#}^3 = 0$. Therefore by using the definition of q, we have:

$$\mathcal{T}_{k,\#} \in \mathcal{L}\Big(H^m_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)\right), H^{q}_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger\right)\Big).$$

Combining this with (2.5.162) yields:

$$\mathcal{T}_{k,\#}(\chi^{2\epsilon}u) \in H^{q}_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger\right). \tag{2.5.163}$$

Moreover, thanks to (2.5.103) and (2.5.104), we can prove $\chi^\epsilon f_2^k \in H^{q}_{0,\Gamma_\mathrm{M}}\left(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger\right)$. Combining this with (2.5.163) conclude the proof of (2.5.160).

Proof of: f_1^k **satisfies the** $\mathcal{P}_q^\infty$ **property**. According to (2.5.101) we take a test function $\psi \in \mathcal{D}\left(\Gamma \times [0,1]^2 \times]0,\infty[\right)$ one periodic in $\hat{x}$ with compact support an we have to prove the existence of a sequence $(f_{1,l}^k)_{l\in\mathbb{Z}^2}$ of elements of $C^q(\Gamma; \mathbb{C}_{d+k}[\hat{\nu}])$ and $\theta \in \mathbb{Z}^2$ such that:

$$\sum_{l\in\mathbb{Z}^2\setminus\{0\}} |l|^{\theta-2} \|f_{1,l}^k\|^2_{C^q(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])} < \infty, \tag{2.5.164}$$

and

$$\langle \mathcal{T}_k u(x_\Gamma;\cdot) - f_2^k(x_\Gamma;\cdot), \psi\rangle = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} \int_{\hat{Y}_\infty} f_{1,l}^k(x_\Gamma;\hat{\nu})\phi_l(x_\Gamma;\hat{x},\hat{\nu})\overline{\psi}(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}, \qquad (2.5.165)$$

where $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\mathbb{H}_{\mathrm{comp}}\left(\hat{Y}_\infty\right)^\dagger - \mathbb{H}_{\mathrm{comp}}\left(\hat{Y}_\infty\right)}$. By using the definition of f_2^k and $\mathcal{T}_k$ and the property:

$$\mathrm{supp}(\psi) \setminus [0,1]^2 \times]0,\infty[,$$

we can prove that $\langle \mathcal{T}_k u_2 - f_2^k, \psi\rangle = 0$. Therefore:

$$\left\langle \mathcal{T}_k u - f_2^k, \psi\right\rangle = \langle \mathcal{T}_k u_1, \psi\rangle\,.$$

Since u_1 satisfies the $\mathcal{P}_m^\infty$ property, there exists a sequence $(u_{1,l})_{l\in\mathbb{Z}^2\setminus\{0\}}$ of element of $C^q(\Gamma;\mathbb{C}_d[\hat{\nu}])$ and $\theta\in\mathbb{Z}$ such that:

$$\sum_{l\in\mathbb{Z}^2\setminus\{0\}} |l|^\theta \|u_{1,l}\|^2_{C^m(\Gamma;\mathbb{C}_d[\hat{\nu}])} < \infty, \qquad (2.5.166)$$

and for all $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma\times\hat{Y}_\infty$:

$$u_1(x_\Gamma;\hat{x},\hat{\nu}) = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} u_{1,l}(x_\Gamma;\hat{\nu})\phi_l(x_\Gamma;\hat{x},\hat{\nu}). \qquad (2.5.167)$$

Now let us prove that for all $x_\Gamma\in\Gamma$:

$$\langle \mathcal{T}_k u_1(x_\Gamma;\cdot), \psi\rangle = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} \Big\langle \mathcal{T}_k\big(\phi_l u_{1,l}\big)(x_\Gamma;\cdot), \psi\Big\rangle. \qquad (2.5.168)$$

According to Proposition 2.5.18, we have in the space:

$$\exp\left(-\pi g_{\min}\hat{\nu}\right) H^m\big(\Gamma; C_b^2([0,1]^2\times[\eta_\psi,\infty[)\big) \text{ with } \eta_\psi := \inf\{\hat{\nu}, \exists\hat{x} \text{ st } (\hat{x},\hat{\nu})\in\mathrm{supp}(\psi)\},$$

the following convergence:

$$u_1 = \lim_{M\to\infty} \sum_{l\in\mathbb{Z}^2\setminus\{0\},|l|\le M} \phi_l u_{1,l}. \qquad (2.5.169)$$

We can prove from the expression of the operator $\mathcal{T}_k$ and the definition of q that:

$$\mathcal{T}_k : \exp\left(-\pi g_{\min}\hat{\nu}\right) H^m\big(\Gamma; C_b^2([0,1]^2\times[\eta_\psi,\infty[)\big) \mapsto H^q\big(\Gamma; C_b^2([0,1]^2\times[\eta_\psi,\infty[)\big).$$

Combining this with (2.5.169) yields the following convergence:

$$\mathcal{T}_k u_1 = \lim_{M\to\infty} \sum_{l\in\mathbb{Z}^2\setminus\{0\},|l|\le M} \mathcal{T}_k(\phi_l u_{1,l}),$$

in the space $H^q\big(\Gamma; C_b^2([0,1]^2\times[\eta_\psi,\infty[)\big)$. Combining this with the compactness of $\mathrm{supp}(\psi)$ conclude the proof of (2.5.168).

Therefore, if we success to construct a sequence $(f_{1,l})_{l\in\mathbb{Z}^2\setminus\{0\}}$ such that:

$$\forall l\in\mathbb{Z}^2\setminus\{0\},\ \ f_{1,l}\phi_l = \mathcal{T}_k\big(\phi_l u_{1,l}\big)$$

then we will conclude the proof of (2.5.165). Therefore let us prove the following result:

Lemma 2.5.21. *For all $l \in \mathbb{Z}^2 \setminus \{0\}$ there exists an operator*

$$\mathcal{T}_l^k \in \mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}]), H^q_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_{d+k}[\hat{\nu}])\right), \tag{2.5.170}$$

such that for all $p_l \in H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}])$ we have:

$$\mathcal{T}_k(\phi_l p_l) = \phi_l \mathcal{T}_l^k(p_l). \tag{2.5.171}$$

Moreover, there exists $C > 0$ such that:

$$\forall l \in \mathbb{Z}^2 \setminus \{0\}, \ \|\mathcal{T}_l^k\|_{\mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]), H^q_{0,\Gamma_\mathrm{M}}\left(\Gamma;\mathbb{C}_{d+p}[\hat{\nu}]\right)\right)} \leq C|l|^2. \tag{2.5.172}$$

To prove Lemma 2.5.21 we need the following intermediate result:
Proposition 2.5.22. *For all $l \in \mathbb{Z}^2 \setminus \{0\}$ there exists:*

$$\begin{cases} \mathcal{T}_{l,\nabla_\Gamma} \in \mathcal{L}\Big(H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}]), (H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]))^3\Big), \\ \mathcal{T}_{l,\mathrm{div}_\Gamma} \in \mathcal{L}\Big((H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}]))^3, H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}])\Big), \\ \mathcal{T}_{l,\widehat{\nabla}} \in \mathcal{L}\Big(H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}]), (H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]))^3\Big), \\ \mathcal{T}_{l,\widehat{\mathrm{div}}} \in \mathcal{L}\Big((H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}]))^3, H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}])\Big), \end{cases} \tag{2.5.173}$$

such that for all $p_l \in H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}])$ and $\vec{p_l} \in (H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_d[\hat{\nu}]))^3$ the following identities hold:

$$\begin{cases} \nabla_\Gamma(p_l\phi_l) = \phi_l \mathcal{T}_{l,\nabla_\Gamma} p_l, & \widehat{\nabla}(p_l\phi_l) = \phi_l \mathcal{T}_{l,\widehat{\nabla}} p_l, \\ \mathrm{div}_\Gamma(\vec{p_l}\phi_l) = \phi_l \mathcal{T}_{l,\mathrm{div}_\Gamma} \vec{p_l}, & \widehat{\mathrm{div}}(\vec{p_l}\phi_l) = \phi_l \mathcal{T}_{l,\widehat{\mathrm{div}}} \vec{p_l}. \end{cases} \tag{2.5.174}$$

Moreover there exists $C > 0$ such that for all $l \in \mathbb{Z}^2$ the following estimates hold:

$$\max \left\{ \begin{array}{l} \left\|\mathcal{T}_{l,\nabla_\Gamma}\right\|_{\mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]),(H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+1}[\hat{\nu}]))^3\right)}, \\ \left\|\mathcal{T}_{l,\widehat{\nabla}}\right\|_{\mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]),(H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+1}[\hat{\nu}]))^3\right)}, \\ \left\|\mathcal{T}_{l,\mathrm{div}_\Gamma}\right\|_{\mathcal{L}\left((H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]))^3,H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+1}[\hat{\nu}])\right)}, \\ \left\|\mathcal{T}_{l,\widehat{\mathrm{div}}}\right\|_{\mathcal{L}\left((H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]))^3,H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+1}[\hat{\nu}])\right)}. \end{array} \right\} \leq C|l|. \tag{2.5.175}$$

Proof. Let $l \in \mathbb{Z}^2 \setminus \{0\}$. We have

$$\nabla_\Gamma \phi_l = -2\pi(\nabla_\Gamma \lambda_l)\hat{\nu}\phi_l \quad \text{and} \quad \widehat{\nabla}\,\phi_l = \begin{pmatrix} \mathrm{D}\,\psi_\Gamma{}^\dagger\, 2\pi i l \\ -2\pi\lambda_l \end{pmatrix} \phi_l.$$

Therefore according to the Leibniz formula, we have for all $p \in H^m_{0,\Gamma_\mathrm{M}}(\Gamma; \mathbb{C}_n[\hat{\nu}])$ that:

$$\nabla_\Gamma(p\phi_l) = \nabla_\Gamma p - 2\pi(\nabla_\Gamma \lambda_l)\hat{\nu}p\phi_l \quad \text{and} \quad \widehat{\nabla}(p\phi_l) = \left(\begin{pmatrix} 0 \\ \partial\hat{\nu} \end{pmatrix} p - \begin{pmatrix} \mathrm{D}\,\psi_\Gamma{}^\dagger\, 2\pi i l \\ 2\pi\lambda_l \end{pmatrix} p \right) \phi_l.$$

Thus the following choice:

$$\mathcal{T}_{l,\nabla_\Gamma} := \nabla_\Gamma - 2\pi(\nabla_\Gamma \lambda_l)\hat{\nu} \quad \text{and} \quad \mathcal{T}_{l,\widehat{\nabla}} := \begin{pmatrix} 0 \\ \partial\hat{\nu} \end{pmatrix} - \begin{pmatrix} \mathrm{D}\,\psi_\Gamma{}^\dagger\, 2\pi i l \\ 2\pi\lambda_l \end{pmatrix},$$

yields the first line of (2.5.174). Furthermore two first lines of (2.5.175) are a direct consequence of Proposition 2.5.20 and Proposition 2.5.13.

For the divergence operators the proof is similar. □

Proof of Proposition 2.5.21. We recall that for all $p \in \mathbb{N}$ we have

$$\mathcal{T}_k(p_l\phi_l) = \sum_{i=0}^{3} \mathcal{T}_{k,i}(p_l\phi_l), \tag{2.5.176}$$

where we defined the following operators for v smooth enough:

$$\begin{cases} \mathcal{T}_{k,0}v := \widehat{\operatorname{div}}\left(\mathcal{C}^{(k)}\hat{\nu}^k\,\widehat{\nabla}\,v\right), & \mathcal{T}_{p,1}v := \operatorname{div}_\Gamma\left(\mathcal{C}^{(k-1)}\hat{\nu}^{k-1}\,\widehat{\nabla}\,v\right), \\ \mathcal{T}_{k,2}v := \widehat{\operatorname{div}}\left(\mathcal{C}^{(k-1)}\hat{\nu}^{k-1}\,\nabla_\Gamma\,v\right), & \mathcal{T}_{p,3}v := \operatorname{div}_\Gamma\left(\mathcal{C}^{(k-2)}\hat{\nu}^{k-2}\,\nabla_\Gamma\,v\right) + k^2c^{((p-2))}\hat{\nu}^{p-2}v. \end{cases}$$

Moreover thanks to (2.5.174), we have:

$$\forall 0 \leq i \leq 3,\ \mathcal{T}_{k,i}(p_l\phi_l) = \mathcal{T}^l_{k,i}(p_l)\phi_l, \tag{2.5.177}$$

where we defined for $v \in H^m(\Gamma;\mathbb{C}_d[\hat{\nu}])$:

$$\begin{cases} \mathcal{T}^l_{k,0}v := \mathcal{T}_{l,\widehat{\operatorname{div}}}\left(\mathcal{C}^{(k)}\hat{\nu}^k\mathcal{T}_{l,\widehat{\nabla}}v\right), & \mathcal{T}^l_{p,1}v := \mathcal{T}_{l,\operatorname{div}_\Gamma}\left(\mathcal{C}^{(k-1)}\hat{\nu}^{k-1}\mathcal{T}_{l,\widehat{\nabla}}v\right), \\ \mathcal{T}^l_{k,2}v := \mathcal{T}_{l,\widehat{\operatorname{div}}}\left(\mathcal{C}^{(k-1)}\hat{\nu}^{k-1}\mathcal{T}_{l,\nabla_\Gamma}v\right), & \mathcal{T}^l_{k,3}v := \mathcal{T}_{l,\operatorname{div}_\Gamma}\left(\mathcal{C}^{(k-2)}\hat{\nu}^{k-2}\mathcal{T}_{l,\nabla_\Gamma}v\right) + k^2c^{(k-2)}\hat{\nu}^{k-2}v. \end{cases}$$

Combining (2.5.176) and (2.5.177) and define $\mathcal{T}^l_k := \sum_{i=0}^{3}\mathcal{T}^l_{k,i}$ conclude the proof of (2.5.171).

Now let us prove (2.5.170) and (2.5.172). Indeed thanks to Proposition 2.5.22, we have for all $0 \leq i \leq 2$:

$$\mathcal{T}^l_{k,i} \in \mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]), H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right), \tag{2.5.178}$$

with the existence of $C > 0$ independent of l such that

$$\|\mathcal{T}^l_{k,i}\|_{\mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]),H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right)} \leq C|l|^2. \tag{2.5.179}$$

Moreover if $k = 1$ then $\mathcal{T}^l_{k,3} = 0$. Combining this with (2.5.178) and (2.5.179) yields

$$k = 1 \Longrightarrow \mathcal{T}^l_k \in \mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]), H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right), \tag{2.5.180}$$

with the existence of $C > 0$ independent of l such that:

$$k = 1 \Longrightarrow \|\mathcal{T}^l_k\|_{\mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]),H^{m-1}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right)} \leq C|l|^2. \tag{2.5.181}$$

If $k \geq 2$ then thanks to Proposition 2.5.22, we have:

$$\mathcal{T}^l_{k,3} \in \mathcal{L}\left(H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}]), H^{m-2}_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right), \tag{2.5.182}$$

with the existence of $C > 0$ independent of l such that:

$$\|\mathcal{T}_{k,3}^l\|_{\mathcal{L}\left(H^m_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_d[\hat{\nu}]),H^{m-2}_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right)} \leq C|l|^2. \tag{2.5.183}$$

Combining (2.5.182) and (2.5.183) with (2.5.178) and (2.5.179) yields:

$$k \geq 2 \Longrightarrow \mathcal{T}_k^l \in \mathcal{L}\left(H^m_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_d[\hat{\nu}]), H^{m-2}_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right), \tag{2.5.184}$$

with the existence of $C > 0$ independent of l such that:

$$k \geq 2 \Longrightarrow \|\mathcal{T}_k^l\|_{\mathcal{L}\left(H^m_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_d[\hat{\nu}]),H^{m-2}_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])\right)} \leq C|l|^2. \tag{2.5.185}$$

Thanks to the definition of q and the implications (2.5.180), (2.5.181), (2.5.184) and (2.5.185) we conclude the proof of (2.5.170) and (2.5.172). □

Lemma 2.5.21 has been proved, we now can apply this result. Therefore for all $l \in \mathbb{Z}^2\backslash\{0\}$, we have

$$\mathcal{T}_k\big(\phi_l u_{1,l}\big) = f_{1,l}\phi_l, \quad \text{with} \quad f_{1,l} := \mathcal{T}_k^l u_{1,l}.$$

Therefore it remains to prove (2.5.164) and $f_{1,l} \in H^q_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])$. Indeed thanks to Lemma 2.5.21 we have: $f_{1,l} \in H^q_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])$ and the existence of $C > 0$ independent of l such that:

$$\|f_{1,l}\|_{H^q_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])} \leq C|l|^2\|u_{1,l}\|_{H^m_{0,\Gamma_{\mathrm{M}}}(\Gamma;\mathbb{C}_{d+k}[\hat{\nu}])}.$$

Combining this with (2.5.166) conclude the proof of (2.5.164).

Proof of (2.5.161). It is a direct consequence of Proposition 2.5.18 and that f_1^k satisfies the $\mathcal{P}_q^\infty$ property.

2.5.2 Matching conditions

Taking $k = 1$ in the matching conditions (2.3.35) implies:

$$\forall x_\Gamma \in \Gamma, \forall 1 \leq n \leq m_\Gamma,\ \partial_{\hat{\nu}} u_{n-1}(x_\Gamma, 0) = p_n^1(x_\Gamma). \tag{2.5.186}$$

Here, we prove that under regularities conditions on our ansatz that (2.5.186) combined with (2.5.106) is equivalent to (2.3.35).

Proposition 2.5.23. *If we have* (2.5.105) *and* (2.5.106) *then for all $2 \leq k' \leq n \leq m_\Gamma$ we have for all $x_\Gamma \in \Gamma$:*

$$p_{k'}^n(x_\Gamma) = -\sum_{i=1}^{k'}\left(\frac{(k'-i)(k'-1)c^{(i)}p_{k'-i}^{n-i} + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\,\nabla_\Gamma\, p_{k'-i}^{n-i}\right) + k^2c^{(i-2)}p_{k'-i}^{n-i}}{k'(k'-1)}\right)(x_\Gamma).$$

Proof. Let $0 \leq n \leq m_\Gamma$. We have seen that the hypothesis (2.5.105) and expression (2.5.106) imply that:

$$\sum_{i=0}^{n}\mathcal{T}_i\hat{u}_{n-i} = 0.$$

Moreover thanks to Proposition 2.5.15, there exist $(A_n, B_n) \in H^{m_\Gamma+\frac{1}{2}-n}\big(\Gamma; \mathbb{C}_{\max(n-2,0)}[\hat{\nu}]\big) \times H^{m_\Gamma+\frac{1}{2}-n}\big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\big)$ such that:

$$\sum_{i=0}^{n} \mathcal{T}_i \hat{u}_{n-i} = A_n + B_n,$$

with:

$$A_n = \sum_{\substack{0\le i\le n\\ 0\le k'\le n-i}} \left(\partial_{\hat{\nu}}\big(c^{(i)}\hat{\nu}^i \partial_{\hat{\nu}}(p_{k'}^{n-i}\hat{\nu}^{k'})\big) + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\nu^{i-2}\,\nabla_\Gamma(p_{k'}^{n-i}\hat{\nu}^{k'})\right) + k^2 c^{(i-2)} p_{k'}^{n-i}\hat{\nu}^{k'}\right).$$

Using Proposition 2.5.9 yields $A_n = 0$. Therefore:

$$\begin{aligned}
0 &= \sum_{\substack{0\le i\le n\\ 0\le k'\le n-i}} \left(\partial_{\hat{\nu}}\big(c^{(i)}\hat{\nu}^i \partial_{\hat{\nu}}(p_{k'}^{n-i}\hat{\nu}^{k'})\big) + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\nu^{i-2}\,\nabla_\Gamma(p_{k'}^{n-i}\hat{\nu}^{k'})\right) + k^2 c^{(i-2)} p_{k'}^{n-i}\hat{\nu}^{k'}\right),\\
&= \sum_{\substack{0\le i\le n\\ 0\le k'\le n-i}} \left(k'(i+k'-1)c^{(i)}\hat{\nu}^{i+k'-2}p_{k'}^{n-i} + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\nu^{i-2}\,\nabla_\Gamma(p_{k'}^{n-i}\hat{\nu}^{k'})\right) + k^2 c^{(i-2)} p_{k'}^{n-i}\hat{\nu}^{k'}\right),\\
&= \sum_{\substack{2\le k'\le n\\ 0\le i\le k'}} \left((k'-i)(k'-1)c^{(i)}p_{k'-i}^{n-i} + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\,\nabla_\Gamma(p_{k'-i}^{n-i})\right) + k^2 c^{(i-2)} p_{k'-i}^{n-i}\right)\hat{\nu}^{k'-2}.
\end{aligned}$$

Therefore, by identifying the coefficient in front of the $\hat{\nu}^{k'-2}$ term with zero, we get for all $2 \le k \le n$.

$$\sum_{i=0}^{k'} \left((k'-i)(k'-1)c^{(i)}p_{k'-i}^{n-i} + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\,\nabla_\Gamma(p_{k'-i}^{n-i})\right) + k^2 c^{(i-2)} p_{k'-i}^{n-i}\right) = 0,$$

which leads to:

$$k'(k'-1)p_{k'}^{n} = -\sum_{i=1}^{k'} \left((k'-i)(k'-1)c^{(i)}p_{k'-i}^{n-i} + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\,\nabla_\Gamma\, p_{k'-i}^{n-i}\right) + k^2 c^{(i-2)} p_{k'-i}^{n-i}\right),$$

and then conclude the proof. □

Proposition 2.5.24. *Let* $0 \le m \le m_\Gamma$ $u \in H^m(\Gamma\times]0,\eta_0[)$ *satisfying* (2.3.32) *then for all* $2 \le k' \le m$ *we have:*

$$\frac{\partial_{\hat{\nu}}^{k'} u}{k'!}(x_\Gamma, 0) = -\sum_{i=1}^{k'} \left(\frac{(k'-i)(k'-1)c^{(i)}\frac{\partial_{\hat{\nu}}^{k'-i}u}{(k'-i)!} + \operatorname{div}_\Gamma\left(\mathcal{C}^{(i-2)}\,\nabla_\Gamma\,\frac{\partial_{\hat{\nu}}^{k'-i}u}{(k'-i)!}\right) + k^2 c^{(i-2)}\frac{\partial_{\hat{\nu}}^{k'-i}u}{(k'-i)!}}{k'(k'-1)}\right)(x_\Gamma, 0).$$

Proof. Let $2 \le k \le m$. We recall that (2.3.32) means:

$$L := \partial\hat{\nu}(\mathrm{C}\,\partial\hat{\nu}u) + \operatorname{div}_\Gamma(\mathcal{C}\,\nabla_\Gamma\, u) + k^2\,\mathrm{C}\,u = 0. \tag{2.5.187}$$

Thanks to the Leibniz formula, we have an explicit expression of $\partial_{\hat\nu}^{k'-2}L$:

$$
\begin{aligned}
\partial_{\hat\nu}^{k'-2}L &= \partial_{\hat\nu}^{k'-1}(\mathrm{C}\,\partial_{\hat\nu}u) + \mathrm{div}_\Gamma\left(\partial_{\hat\nu}^{k'-2}(\mathcal{C}\,\nabla_\Gamma u)\right) + \partial_{\hat\nu}^{k'-2}(k^2\,\mathrm{C}\,u),\\
&= \sum_{l=0}^{k'-1}\binom{k'-1}{l}(\partial_{\hat\nu}^{k'-1-l}\,\mathrm{C})\partial_{\hat\nu}^{l+1}u\\
&+ \sum_{l=0}^{k'-2}\binom{k'-2}{l}\left(\mathrm{div}_\Gamma\left((\partial_{\hat\nu}^{k'-2-l}\,\mathcal{C})\,\nabla_\Gamma(\partial_{\hat\nu}^{l}u)\right) + k^2(\partial_{\hat\nu}^{k'-l-2}\,\mathrm{C})\partial_{\hat\nu}^{l}u\right).
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\partial_{\hat\nu}^{k'-2}L &= \sum_{l=1}^{k'}\binom{k'-1}{l-1}(\partial_{\hat\nu}^{k'-l}\,\mathrm{C})\partial_{\hat\nu}^{l}u\\
&+ \sum_{l=0}^{k'-2}\binom{k'-2}{l}\left(\mathrm{div}_\Gamma\left((\partial_{\hat\nu}^{k'-2-l}\,\mathcal{C})\,\nabla_\Gamma(\partial_{\hat\nu}^{l}u)\right) + k^2(\partial_{\hat\nu}^{k'-l-2}\,\mathrm{C})\partial_{\hat\nu}^{l}u\right),\\
&= \sum_{l=1}^{k'}\frac{(k'-1)!}{(k'-l)!(l-1)!}(\partial_{\hat\nu}^{k'-l}\,\mathrm{C})\partial_{\hat\nu}^{l}u\\
&+ \sum_{l=0}^{k'-2}\frac{(k'-2)!}{(k'-2-l)!l!}\left(\mathrm{div}_\Gamma\left((\partial_{\hat\nu}^{k'-2-l}\,\mathcal{C})\,\nabla_\Gamma(\partial_{\hat\nu}^{l}u)\right) + k^2(\partial_{\hat\nu}^{k'-l-2}\,\mathrm{C})\partial_{\hat\nu}^{l}u\right),\\
&= \frac{(k'-1)!}{(l-1)!}c^{(k'-l)}\partial_{\hat\nu}^{l}u + \frac{(k'-2)!}{l!}\left(\mathrm{div}_\Gamma\,\mathcal{C}^{(k'-2-l)}\,\nabla_\Gamma + k^2c^{(k'-l-2)}\right)\partial_{\hat\nu}^{l}u.
\end{aligned}
$$

We recall for $x_\Gamma \in \Gamma$, the definition (2.3.27) of $c^{(l)}$ and $\mathcal{C}^{(l)}$:

$$
\mathcal{C}^{(k)}(x_\Gamma) := \frac{1}{k!}\partial_\nu^k\,\mathcal{C}(x_\Gamma, 0) \quad \text{if} \quad k \geq 0 \quad \text{else} \quad \mathcal{C}^{(k)}(x_\Gamma) := 0,
$$

and $c^{(k)}(x_\Gamma) := \left(\mathcal{C}^{(k)}(x_\Gamma)\cdot n(x_\Gamma)\right)\cdot n(x_\Gamma)$. Therefore for all $x_\Gamma \in \Gamma$:

$$
\begin{aligned}
\partial_{\hat\nu}^{k'-2}L(x_\Gamma,0) &= \sum_{l=1}^{k'}\frac{l(k'-2)!(k'-1)}{l!}c^{(k'-l)}(x_\Gamma)\partial_{\hat\nu}^{l}u(x_\Gamma,0)\\
&+ \sum_{l=0}^{k'-1}\frac{(k'-2)!}{l!}\left(\mathrm{div}_\Gamma\left(\mathcal{C}^{(k'-2-l)}\,\nabla_\Gamma(\partial_{\hat\nu}^{l}u)\right) + k^2c^{(k'-l-2)}\partial_{\hat\nu}^{l}u\right)(x_\Gamma,0).
\end{aligned}
$$

Combining this with (2.5.187) yields for all x_Γ:

$$
\begin{aligned}
0 &= \sum_{l=1}^{k'} l(k'-1)c^{(k'-l)}(x_\Gamma)\frac{\partial_{\hat\nu}^{l}u}{l!}(x_\Gamma,0)\\
&+ \sum_{l=0}^{k'-1}\left(\mathrm{div}_\Gamma\left(\mathcal{C}^{(k'-2-l)}\,\nabla_\Gamma(\frac{\partial_{\hat\nu}^{l}u}{l!})\right) + k^2c^{(k'-l-2)}\frac{\partial_{\hat\nu}^{l}u}{l!}\right)(x_\Gamma,0).
\end{aligned}
$$

Therefore:

$$k'(k'-1)\frac{\partial_{\hat\nu}^{k'}u}{k'!}(x_\Gamma,0) = -\sum_{l=2}^{k'-1} l(k'-1)c^{(k'-l)}(x_\Gamma)\frac{\partial_{\hat\nu}^{l}u}{l!}(x_\Gamma,0)$$
$$-\sum_{l=0}^{k'-1}\Big(\mathrm{div}_\Gamma\left(\mathcal{C}^{(k'-2-l)}\,\nabla_\Gamma(\frac{\partial_{\hat\nu}^{l}u}{l!})\right)(x_\Gamma,0) + k^2c^{(k'-l-2)}\frac{\partial_{\hat\nu}^{l}u}{l!}(x_\Gamma,0)\Big),$$

which conclude the proof. □

Corollary 2.5.25. *If the regularity conditions* (2.5.105) *hold and* (2.5.106) *is verified then* (2.5.186) *is a sufficient condition for* (2.3.35).

Proof. Define the two sequences $(U_k)_k$ and (V_k) by for k and $x_\Gamma\in\Gamma$ by:

$$U_k(x_\Gamma) := p_k^{n+k}(x_\Gamma) \quad \text{and} \quad V_k(x_\Gamma) := \frac{1}{k!}\partial_\nu^k u_n(x_\Gamma,0).$$

From (2.5.186) we easily get $U_0 = V_0$ and $U_1 = V_1$. Since u_n satisfies the assumption of Proposition 2.5.24 then we have for all $2\le k\le n$:

$$V_k = -\frac{1}{k(k-1)}\sum_{i=1}^{k}\left((k-i)(k-1)c^{((i))} + \mathrm{div}_\Gamma\,\mathcal{C}^{((i-2))}\,\nabla_\Gamma + k^2c^{((i-2))}\right)V_{k-i}.$$

Moreover if the assumption of Proposition 2.5.23 holds then we have for all $2\le k\le n$:

$$U_k = -\frac{1}{k(k-1)}\sum_{i=1}^{k}\left((k-i)(k-1)c^{((i))} + \mathrm{div}_\Gamma\,\mathcal{C}^{((i-2))}\,\nabla_\Gamma + k^2c^{((i-2))}\right)U_{k-i}.$$

Thus U_n and V_n satisfy the same recurrence relation. Therefore these sequence are equal, which conclude the proof. □

2.5.3 Final explicit definition of the ansatz

We have seen that the far and near field sequences are linked via (2.5.186). But these relations do not yet provide an explicit definition of our ansatz. Indeed from this last relation the quantities p_n^1 are found to depend on $\hat u_{n+1}$ and this last relation does not provide an explicit definition of the sequence of far fields $(u_n)_{n\in\mathbb{N}}$. Using Proposition 2.5.11 yields that this last relation becomes:

$$\forall x_\Gamma\in\Gamma,\ \partial_n u_n(x_\Gamma,0) = -\mu\Big(\sum_{i=0}^{n}\left(\mathcal{T}_i\hat u_{n+1-i}\right)(x_\Gamma;\cdot)\Big). \tag{2.5.188}$$

Combining (2.5.188) with (2.5.106) enables us to obtain the following explicit relation:

$$\forall x_\Gamma\in\Gamma,\ \partial_n u_n(x_\Gamma,0) = l_n(x_\Gamma), \tag{2.5.189}$$

where we define for n the following quantity:

$$\begin{cases} l_n(x_\Gamma) := \displaystyle\sum_{i=1}^{n+1}\sum_{j=1}^{n+1-i}\mu\Big(\Big(\mathcal{T}_i\big(\mathcal{T}_0^{-1}(\mathcal{T}_j\hat u_{n+1-i-j})\big)\Big)(x_\Gamma;\cdot)\Big) \\ \qquad -\displaystyle\sum_{i=1}^{n+1}\mu\Big((\mathcal{T}_i u_{n+1-j})(x_\Gamma;\cdot)\Big). \end{cases} \tag{2.5.190}$$

Indeed this last quantity can be computed with the knowledge of $\hat{u}_0 \cdots \hat{u}_{n-1}$. Proposition 2.5.17 and Proposition 2.5.25 require some regularity assumptions on the far field traces and thus it remains to prove that these assumptions hold. Thus let us prove the following result:

Proposition 2.5.26. *Let $-1 \leq n \leq m_\Gamma - 1$ such that (2.5.105) and (2.5.106) are satisfied then we have the regularity*

$$l_{n+1} \in H^{m_\Gamma - n - \frac{1}{2}}(\Gamma), \tag{2.5.191}$$

and then we can define for $n+1$ the far field u_{n+1} as the unique solution of (2.3.32) and (2.3.33) with the boundary conditions:

$$\forall x_\Gamma \in \Gamma,\ \partial_n u_{n+1}(x_\Gamma, 0) = l_{n+1}(x_\Gamma).$$

Moreover this last solution have the regularity $u_{n+1} \in H^{m_\Gamma - n}(\Omega_0)$.

Proof. First prove (2.5.191). We define for for all $1 \leq i \leq n$ and $x_\Gamma \in \Gamma$ the following quantity:

$$Q_i(x_\Gamma; \cdot) := u_{n+1-i}(x_\Gamma, 0) - \sum_{j=1}^{n+1-i} \big((\mathcal{T}_0^{-1}\mathcal{T}_j)\hat{u}_{n+1-i-j}\big)(x_\Gamma; \cdot),$$

in order to have the following rewriting:

$$l_{n+1}(x_\Gamma) = \sum_{i=1}^{n} \mu\Big(T_i Q_i(x_\Gamma; \cdot)\Big). \tag{2.5.192}$$

Thanks to Proposition 2.5.17 we get for all $1 \leq i \leq n$ that the following quantity defined for $x_\Gamma \in \Gamma$: belongs to the space:

$$\forall 1 \leq i \leq n,\ Q_i \in H_{0,\Gamma_M}^{m_\Gamma + \frac{1}{2} - (n+1-i)}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big) \oplus H^{m_\Gamma + \frac{1}{2} - (n+1-i)}\Big(\Gamma; \mathbb{C}_{n+1-i}[\hat{\nu}]\Big).$$

Then using Proposition 2.5.15 yields that for all $2 \leq j \leq n+1-i$:

$$\begin{cases} \mathcal{T}_1 Q_1 \in H_{0,\Gamma_M}^{m_\Gamma + \frac{1}{2} - (n+2-1)}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big) \oplus H^{m_\Gamma + \frac{1}{2} - (n+2-1)}\Big(\Gamma; \mathbb{C}_{n+2-1}[\hat{\nu}]\Big), \\ \mathcal{T}_i Q_i \in H_{0,\Gamma_M}^{m_\Gamma + \frac{1}{2} - (n+3-i)}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big) \oplus H^{m_\Gamma + \frac{1}{2} - (n+3-i)}\Big(\Gamma; \mathbb{C}_{n}[\hat{\nu}]\Big), \end{cases}$$

which leads to:

$$\sum_{i=1}^{n} \mathcal{T}_i Q_i \in H_{0,\Gamma_M}^{m_\Gamma + \frac{1}{2} - (n+1)}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big) \oplus H^{m_\Gamma + \frac{1}{2} - (n+1)}\Big(\Gamma; \mathbb{C}_{n+1}[\hat{\nu}]\Big). \tag{2.5.193}$$

Thus by using that $f \mapsto \mu\Big(f\Big)$ doesn't depend on $x_\Gamma \in \Gamma$, we obtain that combining (2.5.193) and (2.5.192) yields (2.5.191).

The rest of the result is a direct consequence of regularity results for Helmholtz equation (see [60, Theorem 2.5.21], [60, Theorem 2.6.7] and [57, Theorem 4.21]) we can apply because Γ has C^{m_Γ} regularity. $\square$

Thus can state the first main result of this work which is a direct consequence of this last result:

Lemma 2.5.27. *Let $0 \leq n \leq m_\Gamma$. We can define the far and near fields $(\hat{u}_n, u_n)$ with the knowledge of $(\hat{u}_{n'}, u_{n'})_{0\leq n' \leq n-1}$ by:*

- *The near field is defined for $(x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty$ by:*

$$\hat{u}_n(x_\Gamma; \hat{x}, \hat{\nu}) := u_n(x_\Gamma, 0) - \sum_{i=1}^{n} \big(\mathcal{T}_0^{-1}(\mathcal{T}_i \hat{u}_{n-i})\big)(x_\Gamma; \hat{x}, \hat{\nu}).$$

- *The far field is defined by the unique solution of: Find $u_n \in H^1(\Omega_0)$ satisfying (2.3.32), (2.3.33) and*

$$\partial_{\hat{\nu}} u_n(x_\Gamma, 0) := l_n(x_\Gamma), \tag{2.5.194}$$

  where l_n is defined by (2.5.190).

We have that:

- *The near field $(\hat{u}_n)_{0\leq n\leq m_\Gamma}$ satisfies (2.3.31).*
- *For all $0 \leq n \leq m_\Gamma$, there exist $(p_n, r_n) \in H^{m_\Gamma+\frac{1}{2}-n}_{0,\Gamma_\mathrm{M}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big) \times H^{m_\Gamma+\frac{1}{2}-n}\Big(\Gamma; \mathbb{C}_n[\hat{\nu}]\Big)$ such that:*

$$u_n = p_n + r_n.$$

- *For all $0 \leq n \leq m_\Gamma$, we have:*

$$u_n \in H^{m_\Gamma+1-n}(\Omega_0). \tag{2.5.195}$$

- *The matching conditions (2.3.35) holds.*

Chapter 3

Theoretical justification of the asymptotic expansion

The existence of $(u_i, \hat{u}_i)_{i\in\mathbb{N}}$ of the formal asymptotic expansion has been proved for $0 \leq n \leq m_\Gamma$ (see Lemma 2.5.27). We therefore can construct the far field

$$u_{n,\delta} := u_0 + \cdots + \delta^n u_n$$

and the near field

$$\hat{u}_{n,\delta} := \hat{u}_0 + \cdots + \delta^n \hat{u}_n.$$

Then we construct a global function on Ω_δ:

$$u^n_{\eta,\delta} := (1-\chi_\eta)u^n_\delta + \chi_\eta\,\mathcal{I}_\delta\,\hat{u}_{n,\delta}, \tag{3.0.1}$$

where $\chi_\eta(x) := \chi(\frac{x_\nu}{\eta})$ and χ is a regular cutoff truncate function such that for all $s \in \mathbb{R}$, $\chi(s) = 1$ if $s \leq 1$ and $\chi(s) = 0$ is $s \geq 2$.

Here we prove that $u^n_{\eta,\delta}$ converges to the exact solution u_δ and prove estimates of the error convergence rate in terms of δ. Although we strongly draw on [4, 3, 5, 6, 12, 18] for the formal construction, we do not use the two-scale convergence method (See [6]). We have inspired by [37, 34, 35]. For this part, we proceed as follow:

- To give a sense of the quantity $\epsilon(\delta,\eta)$ we need to prove that $u^n_{\eta,\delta} \in H^1(\Omega_\delta)$. A sufficient condition is to prove some continuity property of the operator $\mathcal{I}_\delta$ and this continuity property is not trivial.

- We prove that

$$\|v_\delta\|_{H^1(\Omega_\delta)} \leq C \sup_{\phi\in H^1(\Omega_\delta)} \frac{a_\delta(v_\delta,\phi)}{\|\phi\|_{H^1(\Omega_\delta)}}, \forall u_\delta \in H^1\left(\Omega_\delta\right).$$

- We prove estimate of a kind of consistent error:

$$\epsilon(\delta,\eta) := \sup_{\|v\|_{H^1(\Omega_\delta)}=1} a_{\delta,\eta}(u^n_{\eta,\delta} - u_\delta, v).$$

  This quantity measure how $u^n_{\eta,\delta}$ fails to satisfies our problem.

- We deduce that:

$$\|u^n_{\eta,\delta} - u_\delta\|_{H^1(\Omega_\delta)} \leq C\epsilon(\delta,\eta).$$

This procedure will yields a result of justification and error estimate Theorem 3.4.1. This result requires strong assumption on the regularity on our manifold Γ. The error estimate are not valid for point near the surface Γ because there is an phenomenon of boundary layer. We emphasize that there is the same phenomenon in [37, 34, 35].

3.1 Continuity properties of the scaling operator $\mathcal{I}_\delta$

Here a is a large parameter that will be later replaced by η/δ. $C > 0$ is a generic constant independent of a and δ. Introduce the following space:

$$L^2_\#\big(\hat{Y}_a\big) := \left\{u \in L^2_{\mathrm{loc}}(\hat{\Omega}_a),\ \text{one periodic on } \hat{x},\ \|u\|^2_{L^2_\#\left(\hat{Y}_a\right)} := \int_{Y^a} |u(\hat{x},\hat{\nu})|^2 d\hat{x}d\hat{\nu} < \infty\right\},$$

where $\hat{Y}_a :=]0,1[^2\times]-1,a[$ and $\hat{\Omega}_a := \mathbb{R}^2\times]-1,a[$. We also introduce for $\epsilon > 0$ the set $\Omega_{\delta,\epsilon} := \Gamma\times]-\delta,\epsilon[$.

Proposition 3.1.1. *Let $q > 1$. The operator $\mathcal{I}_\delta$ satisfies:*

$$\mathcal{I}_\delta : H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\big(\hat{Y}_a\big)\right) \mapsto L^2(\Omega_{\delta,\delta a}),$$

and for all $u \in H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\big(\hat{Y}_a\big)\right)$ we have:

$$\|\mathcal{I}_\delta u\|_{L^2(\Omega_{\delta,\delta a})} \leq C\delta^{\frac{1}{2}}\|u\|_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\left(\hat{Y}_a\right)\right)}. \tag{3.1.2}$$

Proof. Let $u \in H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\big(\hat{Y}_a\big)\right)$. The following estimate is trivial:

$$\|\mathcal{I}_\delta u\|_{L^2\left(\Omega_{\delta,\delta a}\right)} \leq \|\mathcal{I}_\delta u\|_{L^2((\Gamma\setminus\Gamma_{\mathrm{M}})\times]-\delta,a\delta[)} + \|\mathcal{I}_\delta u\|_{L^2(\Gamma_{\mathrm{M}}\times]-\delta,a\delta[)}. \tag{3.1.3}$$

Estimate of $\|\mathcal{I}_\delta u\|_{L^2((\Gamma\setminus\Gamma_{\mathrm{M}})\times]-\delta,a\delta[)}$. We have on $\Gamma\setminus\Gamma_{\mathrm{M}}$ that $u(x_\Gamma,\hat{x},\hat{\nu}) = U(x_\Gamma,\hat{\nu})$ for some U in $H^q\big(\Gamma; L^2(]-1,a[)\big)$. Therefore using the change of variable formula yields:

$$\begin{aligned}\|\mathcal{I}_\delta u\|^2_{L^2(\Gamma\setminus\Gamma_{\mathrm{M}}\times]-\delta,\delta a[)} &= \int_{\Gamma\setminus\Gamma_{\mathrm{M}}\times]-\delta,\delta a[} \left|U\left(x_\Gamma,\frac{\nu}{\delta}\right)\right|^2 dx_\Gamma d\nu,\\ &= \delta\int_{\Gamma\setminus\Gamma_{\mathrm{M}}\times]-1,a[} |U(x_\Gamma,\hat{\nu})|^2 dx_\Gamma d\hat{\nu},\end{aligned}$$

which yields:

$$\|\mathcal{I}_\delta u\|_{L^2(\Gamma\setminus\Gamma_{\mathrm{M}}\times]-\delta,\delta a[)} \leq \delta^{\frac{1}{2}}\cdot\|u\|_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\left(\hat{Y}_a\right)\right)}. \tag{3.1.4}$$

Estimate of $L^2\big(\Gamma_{\mathrm{M}}\times]-\delta,a\delta[\big)$. Let us prove that:

$$\|\mathcal{I}_\delta u\|_{L^2(\Gamma_{\mathrm{M}}\times]-\delta,\delta a[)} \leq \delta^{\frac{1}{2}}\cdot\|u\|_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\left(\hat{Y}_a\right)\right)}. \tag{3.1.5}$$

Indeed, according to the definition of the set Γ_{M} and the inverse function theorem applied to ψ_Γ, there exist for all $x_\Gamma \in \Gamma_{\mathrm{M}}$ an open subset $\omega_{x_\Gamma} \subset \mathbb{R}^2$ and $\Gamma_{x_\Gamma} \subset \Gamma_{\mathrm{M}}$ with $x_\Gamma \in \Gamma_{x_\Gamma}$ such that $\psi_\Gamma : \Gamma_{x_\Gamma} \mapsto \omega_{x_\Gamma}$ is a diffeomorphism. Thus we have the following open covers:

$$\Gamma_{\mathrm{M}} \subset \bigcup_{x_\Gamma \in \Gamma_{\mathrm{M}}} \Gamma_{x_\Gamma},$$

and using the compactness of $\overline{\Gamma_{\mathrm{M}}}$ yields the existence of $N \in \mathbb{N}$ and $x_\Gamma^1, \cdots, x_\Gamma^N$ such that :

$$\Gamma_{\mathrm{M}} \subset \bigcup_{i=1}^{N} \Gamma_{x_\Gamma^i}.$$

Therefore we have:

$$\| \mathcal{I}_\delta \, u\|^2_{L^2(\Gamma_{\mathrm{M}} \times]-\delta,\delta a[)} \leq \sum_{i=1}^{N} \| \mathcal{I}_\delta \, u\|^2_{L^2\left(\Gamma_{x_\Gamma^i} \times]-\delta,\delta a[\right)} \leq \sum_{i=1}^{N} \int_{\Gamma_{x_\Gamma^i} \times]-\delta,\delta a[} \left| u\left(x_\Gamma; \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right)\right|^2 dx_\Gamma d\nu,$$

and then:

$$\| \mathcal{I}_\delta \, u\|^2_{L^2(\Gamma_{\mathrm{M}} \times]-\delta,\delta a[)} \leq C \sum_{i=1}^{N} \int_{\Gamma_{x_\Gamma^i} \times]-\delta,\delta a[} \sup_{y_\Gamma \in \Gamma_{\mathrm{M}}} \left\| u\left(y_\Gamma; \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right)\right\|^2 dx_\Gamma d\nu. \tag{3.1.6}$$

Moreover, since $q > 1$, Sobolev embedding results imply $H^q(\Gamma) \subset L^\infty(\Gamma)$ with continuous injection. Therefore for all $x_\Gamma \in \Gamma_{\mathrm{M}}$ we have:

$$\sup_{y_\Gamma \in \Gamma_{\mathrm{M}}} \left\| u\left(y_\Gamma; \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right)\right\|^2 = \left\| u\left(.; \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right)\right\|^2_{L^\infty(\Gamma)} \leq \hat{N}\left(\frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right),$$

where we defined the function $\hat{N} : \hat{\Omega} \mapsto \mathbb{R}^+$ for $(\hat{x}, \hat{\nu}) \in \hat{\Omega}$ by $\hat{N}(\hat{x}, \hat{\nu}) := \|u\,(.; \hat{x}, \hat{\nu})\|^2_{H^q(\Gamma)}$. Combining this with (3.1.6) yields:

$$\| \mathcal{I}_\delta \, u\|^2_{L^2(\Gamma_{\mathrm{M}} \times]-\delta,\delta a[)} \leq C \sum_{i=1}^{N} I_i, \tag{3.1.7}$$

where we defined for $1 \leq i \leq N$:

$$I_i := \int_{\Gamma_{x_\Gamma^i} \times]-\delta,\delta a[} \hat{N}\left(\frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right) dx_\Gamma d\nu. \tag{3.1.8}$$

Let $1 \leq i \leq N$ and let us now estimate I_i. We introduce the map $\phi_i^\delta : \Gamma_{x_\Gamma^i} \times]-\delta, \delta a[\mapsto \left(\omega_{x_\Gamma^i}/\delta\right) \times]-1, a[$ defined for $(x_\Gamma, \nu) \in \Gamma_{x_\Gamma^i} \times]-\delta, \delta a[$ by:

$$\phi_i^\delta(x_\Gamma) := \left(\frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right) \quad \text{with} \quad \left(\omega_{x_\Gamma^i}/\delta\right) := \left\{\frac{x}{\delta}, x \in \omega_{x_\Gamma^i}\right\}.$$

Since we have seen that $\psi_\Gamma : \Gamma_{x_\Gamma^i} \mapsto \omega_{x_\Gamma^i}$ is a diffeomorphism, we have:

$$\forall x_\Gamma \in \Gamma, \; \det\left(\mathrm{D}\,\phi_i^\delta(x_\Gamma)^{-\dagger}\,\mathrm{D}\,\phi_i^\delta(x_\Gamma)^{-1}\right)^{\frac{1}{2}} \leq C\delta^2. \tag{3.1.9}$$

Thanks to the definition of ϕ_i^δ, we get the following rewriting of (3.1.8):

$$I_i = \int_{\Gamma_{x_\Gamma^i} \times]-\delta,\delta a[} (\hat{N} \circ \phi_i^\delta)(x_\Gamma, \nu) dx_\Gamma d\nu.$$

According to the change of variable formula and (3.1.9), we have:

$$I_i \leq C\delta^2 \int_{\omega_{x_\Gamma^i}/\delta \times]-1,a[} \hat{N}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}. \tag{3.1.10}$$

Let L be an arbitrary number, large enough in order to have for all $1 \leq i \leq N$ the inclusion $\omega_{x_\Gamma^i} \subset]0, L[$. Thus (3.1.10) becomes:

$$I_i \leq C\delta^2 \int_{]-N_L, N_L[^2 \times]-1,a[} \hat{N}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}, \tag{3.1.11}$$

where $N_L := 1 + \mathrm{Ent}(\frac{L}{\delta})$ and Ent is the floor function. Using periodicity on $\hat{x}$ of the function $(\hat{x}, \hat{\nu}) \mapsto N(\hat{x}, \hat{\nu})$ yields:

$$\int_{]-N_L, N_L[^2 \times]-1,a[} \hat{N}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = 4N_L^2 \cdot \int_{\hat{Y}_a} \hat{N}(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}. \tag{3.1.12}$$

Moreover, from the definition of the map $\hat{N}$, we have:

$$\int_{\hat{Y}_a} N(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \int_{\hat{Y}_a} \|u\,(.; \hat{x}, \hat{\nu})\|^2_{H^q(\Gamma)} d\hat{x} d\hat{\nu}. \tag{3.1.13}$$

According Fubini theorem and the definition of the space $H^q_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; L^2_\#\big(\hat{Y}_a\big)\Big)$, we have:

$$\int_{\hat{Y}_a} \|u\,(.; \hat{x}, \hat{\nu})\|^2_{H^q(\Gamma)} d\hat{x} d\hat{\nu} = \|u\|^2_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\left(\hat{Y}_a\right)\right)}.$$

Combining this with (3.1.13), (3.1.12) and (3.1.11) yields the following estimate:

$$I_i \leq C\delta \|u\|^2_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\left(\hat{Y}_a\right)\right)}.$$

Combining this with (3.1.7) conclude the proof of (3.1.5). Combining (3.1.4) and (3.1.5) with (3.1.3) conclude the proof. □

To state Corollary 3.1.2, we need to introduce the following space:

$$\mathbb{H}\big(\hat{Y}_a\big) := \left\{ u \in L^2_\#\big(\hat{Y}_a\big),\ \nabla u \in, \big(L^2_\#\big(\hat{Y}_a\big)\big)^3 \right\},$$

and the norm of this space is defined for $u \in \mathbb{H}\big(\hat{Y}_a\big)$ by $\|u\|_{\mathbb{H}\left(\hat{Y}_a\right)} := \|u\|_{L^2(\hat{Y}_a)} + \|\nabla u\|_{L^2(\hat{Y}_a)^3}$.

Corollary 3.1.2. *Let q. For all $\delta > 0$ the operator $\mathcal{I}_\delta$ satisfies:*

$$\mathcal{I}_\delta : H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big) \mapsto H^1(\Omega_{\delta,\delta a}).$$

For all $u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big)$ we have:

$$\| \mathcal{I}_\delta u\|_{\left(H^1(\Omega_{\delta,\delta a})\right)^\dagger} \leq C\delta^{-\frac{1}{2}} \|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_a\right)\right)}. \tag{3.1.14}$$

and

$$\nabla_{\mathcal{L}} \mathcal{I}_\delta u = \mathcal{I}_\delta \Big(\nabla_\Gamma u + \delta^{-1} \widehat{\nabla} u\Big). \tag{3.1.15}$$

Proof. Let $u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big)$ and let us prove (3.1.15). Let $\phi \in \mathcal{D}\big(\Omega_{\delta,\delta a}\big)^3$ and let us prove that:

$$-\int_{\Omega_{\delta,\delta a}} \mathcal{I}_\delta u \mathrm{div}_{\mathcal{L}}(\overline{\phi}) \mathrm{d}x_\Gamma \mathrm{d}\nu = \int_{\Omega_{\delta,\delta a}} \mathcal{I}_\delta \Big(\nabla_\Gamma u + \delta^{-1} \widehat{\nabla} u\Big) \overline{\phi} \mathrm{d}x_\Gamma \mathrm{d}\nu. \tag{3.1.16}$$

We can prove that $C^{q+1}\big(\Gamma \times \overline{\hat{Y}_a}\big) \cap H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big)$ is dense in the space $H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big)$. Therefore there exist, $(u^n)_{n\in\mathbb{N}} \in C^{q+1}\big(\Gamma \times \overline{\hat{Y}_a}\big) \cap H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big)$ such that:

$$u^n \xrightarrow[H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_a\right)\right)]{} u \tag{3.1.17}$$

Let $n \in \mathbb{N}$, since u^n is smooth enough, we can apply Proposition 2.3.1 to u^n. Therefore for all $n \in \mathbb{N}$, we have:

$$\nabla_{\mathcal{L}} \mathcal{I}_\delta u^n = \mathcal{I}_\delta \Big(\nabla_\Gamma u^n_\Gamma + \delta^{-1} \widehat{\nabla} u^n\Big),$$

which leads to:

$$-\int_{\Omega_{\delta,\delta a}} \mathcal{I}_\delta u^n \mathrm{div}_{\mathcal{L}}(\overline{\phi}) \mathrm{d}x_\Gamma \mathrm{d}\nu = \int_{\Omega_{\delta,\delta a}} \mathcal{I}_\delta \Big(\nabla_\Gamma u^n + \delta^{-1} \widehat{\nabla} u^n\Big) \overline{\phi} \mathrm{d}x_\Gamma \mathrm{d}\nu. \tag{3.1.18}$$

Moreover (3.1.17) imply:

$$\nabla_\Gamma u^n + \delta^{-1} \widehat{\nabla} u^n \xrightarrow[H^{q}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;(L^2_{\#}\left(\hat{Y}_a\right))^3\right)]{} \nabla_\Gamma u + \delta^{-1} \widehat{\nabla} u$$

Thus using Proposition 3.1.1 implies that the sequences $\mathcal{I}_\delta u^n$ and $\mathcal{I}_\delta \Big(\nabla_\Gamma u^n + \delta^{-1} \widehat{\nabla} u^n\Big)$ converge in $L^2(\Omega_{\delta,\delta a})$ to $\mathcal{I}_\delta u$ and $\mathcal{I}_\delta \Big(\nabla_\Gamma u + \delta^{-1} \widehat{\nabla} u\Big)$ respectively. Thus we have:

$$-(\mathcal{I}_\delta u, \mathrm{div}_{\mathcal{L}}\phi)_{L^2(\Omega_{\delta,\delta a})} = \Big(\mathcal{I}_\delta \Big(\nabla_\Gamma u + \delta^{-1} \widehat{\nabla} u\Big), \phi\Big)_{L^2(\Omega_{\delta,\delta a})},$$

which conclude the proof of (3.1.15).

Thanks to (3.1.15) we directly get that $\mathcal{I}_\delta\, u \in H^1(\Omega_{\delta,\delta a})$. Now let us prove the estimate (3.1.14). From the definition of $H^{q+1}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_a\big)\Big)$ we have:

$$\nabla_\Gamma\, u + \delta^{-1}\, \widehat{\nabla}\, u \in \Big(H^q_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; L^2_{\#}\big(\hat{Y}_a\big)\Big)\Big)^3.$$

Therefore, according to Proposition 3.1.1 we have:

$$\mathcal{I}_\delta\left(\nabla_\Gamma\, u + \delta^{-1}\, \widehat{\nabla}\right) \in L^2(\Omega_{\delta,\delta a}),$$

with the existence of $C > 0$ independent of u and δ such that:

$$\left\|\mathcal{I}_\delta\left(\nabla_\Gamma\, u + \delta^{-1}\, \widehat{\nabla}\right)\right\|_{L^2(\Omega_{\delta,\delta a})} \leq C\delta^{\frac{1}{2}} \left\|\nabla_\Gamma\, u + \delta^{-1}\, \widehat{\nabla}\, u\right\|_{\left(H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_{\#}\left(\hat{Y}_a\right)\right)\right)^3}.$$

From this equality we get that $\mathcal{I}_\delta\, u$ belongs to $H^1(\Omega_{\delta,\delta a})$ and we have thanks to the estimation which appear in Proposition 3.1.1:

$$\begin{aligned}
\|\,\mathcal{I}_\delta\, u\|_{H^1(\Omega_{\delta,\delta a})} &= \|\,\mathcal{I}_\delta\, u\|_{L^2(\Omega_{\delta,\delta a})} + \left\|\mathcal{I}_\delta \cdot \left(\nabla_\Gamma\, u_\Gamma + \delta^{-1}\, \widehat{\nabla}\, u\right)\right\|_{\Omega_{\delta,\delta a}} \\
&\leq C\|u\|_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_{\#}\left(\hat{Y}_a\right)\right)} + \left\|\nabla_\Gamma\, u_\Gamma + C\delta^{-1}\, \widehat{\nabla}\, u\right\|_{H^q_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; (L^2_{\#}\left(\hat{Y}_a\right))^3\right)} \\
&\leq \delta^{-1}\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_{\#}\left(\hat{Y}_a\right)\right)}.
\end{aligned}$$

Thus the proof is finished. □

To state Corollary 3.1.3 we need to introduce the two following spaces:

- $\mathbb{H}_0\big(\hat{Y}_a\big) \subset \mathbb{H}\big(\hat{Y}_a\big)$ is the space of function $u \in \mathbb{H}\big(\hat{Y}_a\big)$ such that $u = 0$ on $]0,1[^2 \times \{a\}$. The norm of this last space is defined for $u \in \mathbb{H}_0\big(\hat{Y}_a\big)$ by:

$$\|u\|^2_{\mathbb{H}_0\left(\hat{Y}_a\right)} := \int_{Y^a} |\nabla u(\hat{x}, \hat{\nu})|^2 d\hat{x} d\hat{\nu}. \tag{3.1.19}$$

- $H^{1,0}(\Omega_{\delta,\delta a})$ is the space of function $u \in H^1(\Omega_{\delta,\delta a})$ such that $u = 0$ on $\Gamma \times \{\delta a\}$. The norm in this space is defined for $u \in H^{1,0}(\Omega_{\delta,\delta a})$ by:

$$\|u\|^2_{H^{1,0}(\Omega_{\delta,\delta a})} := \int_{\Omega_{\delta,\delta a}} |\,\nabla_{\mathcal{L}}\, u|^2 \mathrm{d}x_\Gamma \mathrm{d}\nu. \tag{3.1.20}$$

For the sequel we identify $L^2(\Omega_{\delta,\delta a})$ as a subset of $H^{1,0}(\Omega_{\delta,\delta a})^\dagger$ with the following canonical injection:

$$\langle u, v\rangle_{H^{1,0}(\Omega_{\delta,\delta a})^\dagger - H^{1,0}(\Omega_{\delta,\delta a})} := \int_{\Omega_{\delta,\delta a}} u\overline{v} \mathrm{d}x_\Gamma \mathrm{d}\nu.$$

Thanks to the Cauchy Schwartz inequality, we can prove:

$$\|u\|_{H^{1,0}(\Omega_{\delta,\delta a})^\dagger} \leq C\|u\|_{L^2(\Omega_{\delta,\delta a})}. \tag{3.1.21}$$

Hereafter

$$\operatorname{div}_{\mathcal{L}} : L^2(\Omega_{\delta,\delta a})^3 \mapsto H^{1,0}(\Omega_{\delta,\delta a})^\dagger, \tag{3.1.22}$$

is the adjoint of the operator $-\nabla_{\mathcal{L}} : H^{1,0}(\Omega_{\delta,\delta a}) \mapsto \left(L^2(\Omega_{\delta,\delta a})\right)^3$ and we have:

$$\|\operatorname{div}_{\mathcal{L}}\|_{\mathcal{L}\left(L^2(\Omega_{\delta,\delta a})^3, H^{1,0}(\Omega_{\delta,\delta a})^\dagger\right)} \leq C. \tag{3.1.23}$$

Corollary 3.1.3. *Let $q > 1$. For all $\delta > 0$ the operator $\mathcal{I}_\delta$ satisfies:*

$$\mathcal{I}_\delta : H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}_0\big(\hat{Y}_a\big)^\dagger\right) \mapsto \left(H^{1,0}(\Omega_{\delta,\delta a})\right)^\dagger.$$

For all $u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\big(\hat{Y}_a\big)\right)$ we have

$$\|\,\mathcal{I}_\delta\, u\|_{\left(H^{1,0}(\Omega_{\delta,\delta a})\right)^\dagger} \leq C\delta^{\frac{3}{2}} \|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}. \tag{3.1.24}$$

Moreover this extension satisfies for all $\mathbf{u} \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; (L^2_\#\big(\hat{Y}_a\big))^3\right)$:

$$\operatorname{div}_{\mathcal{L}} \mathcal{I}_\delta\, \mathbf{u} = \mathcal{I}_\delta\left(\operatorname{div}_\Gamma \mathbf{u}_\Gamma + \delta^{-1}\, \widehat{\operatorname{div}}\, \mathbf{u}\right). \tag{3.1.25}$$

Proof. Let us summarize our proof:

1. We define a map $\tilde{\mathcal{I}}_\delta : H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}_0\big(\hat{Y}_a\big)^\dagger\right) \mapsto \left(H^{1,0}(\Omega_{\delta,\delta a})\right)^\dagger$ such that:

$$\forall u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\big(\hat{Y}_a\big)\right),\ \|\tilde{\mathcal{I}}_\delta u\|_{\left(H^{1,0}(\Omega_{\delta,\delta a})\right)^\dagger} \leq C\delta^{\frac{3}{2}} \|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}. \tag{3.1.26}$$

2. We prove that this map is well extension of $\mathcal{I}_\delta$ in the sense that:

$$\forall u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#\big(\hat{Y}_a\big)\right),\ \tilde{\mathcal{I}}_\delta u = \mathcal{I}_\delta\, u. \tag{3.1.27}$$

3. We prove that:

$$\forall \mathbf{u} \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; (L^2_\#\big(\hat{Y}_a\big))^3\right),\ \operatorname{div}_{\mathcal{L}} \mathcal{I}_\delta\, \mathbf{u} = \mathcal{I}_\delta\left(\operatorname{div}_\Gamma \mathbf{u}_\Gamma + \delta^{-1}\, \widehat{\operatorname{div}}\, \mathbf{u}\right). \tag{3.1.28}$$

Definition of $\tilde{\mathcal{I}}_\delta$ and proof of estimate (3.1.26). To define $\tilde{\mathcal{I}}_\delta$, we first need to introduce the operator $S_a : H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}_0\big(\hat{Y}_a\big)^\dagger\right) \mapsto H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}_0\big(\hat{Y}_a\big)\right)$ defined for $v \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}_0\big(\hat{Y}_a\big)^\dagger\right)$ and $x_\Gamma \in \Gamma$ by:

$$S_a u(x_\Gamma; \cdot) := V,$$

where V is the the unique solution of: Find $V \in \mathbb{H}_0\big(\hat{Y}_a\big)$ such that for all $\phi \in \mathbb{H}_0\big(\hat{Y}_a\big)$:

$$-\int_{\hat{Y}_a} \widehat{\nabla}\, V(x_\Gamma; \hat{x}, \hat{\nu}) \cdot \widehat{\nabla}\, \phi(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \langle u(x_\Gamma; \cdot), \phi \rangle_{\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger - \mathbb{H}_0\left(\hat{Y}_a\right)}. \tag{3.1.29}$$

By using 2.5.13 we can prove that we well have for all $u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\big(\hat{Y}_a\big)^\dagger\right)$:

$$S_a u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\big(\hat{Y}_a\big)\right), \tag{3.1.30}$$

and according to the definition of the norm of the space $\mathbb{H}_0\big(\hat{Y}_a\big)$ given in (3.1.19), we have:

$$\left\|\widehat{\nabla}(S_a u)\right\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\left(\hat{Y}_a\right)\right)} \leq C\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}. \tag{3.1.31}$$

We now prove that we can define $\tilde{\mathcal{I}}_\delta$ for $u \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\big(\hat{Y}_a\big)^\dagger\right)$ by:

$$\tilde{\mathcal{I}}_\delta u := \delta\left(\operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big) - \mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right)\right)\right). \tag{3.1.32}$$

Indeed, by construction of S_a and thanks to (3.1.30) we have:

$$\widehat{\nabla}(S_a u) \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\big(\hat{Y}_a\big)\right)^3 \quad \text{and} \quad \operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right) \in H^{q}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\big(\hat{Y}_a\big)\right). \tag{3.1.33}$$

Thanks to (3.1.31) we have:

$$\|\widehat{\nabla}(S_a u)\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\left(\hat{Y}_a\right)\right)^3} \leq C\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}, \tag{3.1.34}$$

and:

$$\|\operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right)\|_{H^{q}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;L^2_\#\left(\hat{Y}_a\right)\right)} \leq C\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}. \tag{3.1.35}$$

Thus, thanks to (3.1.33), we can apply Proposition 3.1.1, which leads to:

$$\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right) \in L^2(\Omega_{\delta,\delta a}) \quad \text{and} \quad \mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right)\right) \in L^2(\Omega_{\delta,\delta a}), \tag{3.1.36}$$

and the estimate (3.1.34) and (3.1.35) leads to:

$$\|\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\|_{L^2(\Omega_{\delta,\delta a})} \leq C\delta^{\frac{1}{2}}\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)},$$

and

$$\left\|\mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right)\right)\right\|_{L^2(\Omega_{\delta,\delta a})} \leq C\delta^{\frac{1}{2}}\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}. \tag{3.1.37}$$

Thanks to $L^2(\Omega_{\delta,\delta a}) \subset H^{1,0}(\Omega_{\delta,\delta a})^\dagger$ and (3.1.22), (3.1.36) leads to:

$$\left(\mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right)\right), \operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big)\right) \in \left(H^{1,0}(\Omega_{\delta,\delta a})^\dagger\right)^2. \tag{3.1.38}$$

Moreover combining (3.1.21) and (3.1.23), with the estimate (3.1.37) yields:

$$\left\|\operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big)\right\|_{H^{1,0}(\Omega_{\delta,\delta a})^\dagger} \leq C\delta^{\frac{1}{2}}\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}, \tag{3.1.39}$$

and

$$\left\|\mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\widehat{\nabla}(S_a u)\right)\right)\right\|_{H^{1,0}(\Omega_{\delta,\delta a})^\dagger} \leq C\delta^{\frac{1}{2}}\|u\|_{H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}_0\left(\hat{Y}_a\right)^\dagger\right)}. \tag{3.1.40}$$

Thanks to (3.1.38), the quantity $\tilde{\mathcal{I}}_\delta u$ can well be defined by (3.1.32) and belongs to $H^{1,0}(\Omega_{\delta,\delta a})^\dagger$.

Combining (3.1.32) with the estimate (3.1.39) and (3.1.40) conclude the proof of the estimate (3.1.26).

Proof of (3.1.27). Thanks to (3.1.26) and Proposition 3.1.1, we have:

$$(\mathcal{I}_\delta, \tilde{\mathcal{I}}_\delta) \in \mathcal{L}\left(H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; L^2_\#(\hat{Y}_a)\right), H^{1,0}(\Omega_{\delta,\delta a})^\dagger\right)^2. \tag{3.1.41}$$

Moreover, we can prove that the following space:

$$C^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; C^\infty_{\mathrm{per,a}}\right) \quad \text{with} \quad C^\infty_{\mathrm{per,a}} := C^\infty(\overline{\hat{\Omega}_a}) \cap \mathbb{H}_0(\hat{Y}_a)$$

is dense into $H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}_0(\hat{Y}_a)^\dagger\right)$. Therefore it remains to prove (3.1.27) for $u \in C^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; C^\infty_{\mathrm{per,a}}\right)$.

Regularity results for elliptic operator (see [60, Theorem 2.5.21], [60, Theorem 2.6.7] and [57, Theorem 4.21]) lead to $S_a u \in C^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; C^\infty_{\mathrm{per,a}}\right)$. Thus we have:

$$\widehat{\nabla}(S_a u) \in C^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; C^\infty_{\mathrm{per,a}}\right) \quad \text{and} \quad (\widehat{\nabla}(S_a u)).n = 0 \text{ on } \Gamma\times]0,1[^2\times\{-1\}. \tag{3.1.42}$$

Thus, we can apply Proposition 2.3.2 which leads to:

$$\operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big) = \mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\left(\widehat{\nabla}(S_a u)\right)_\Gamma\right) + \delta^{-1}\,\widehat{\operatorname{div}}\left(\widehat{\nabla}(S_a u)\right)\right) \text{ in } L^2(\Omega_{\delta,\delta a}). \tag{3.1.43}$$

From the boundary conditions that appear in (3.1.42) we get $(\mathcal{I}_\delta\, v).n = 0$ on $\Gamma \times \{-\delta\}$. Therefore (3.1.43) becomes:

$$\operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big) = \mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\left(\widehat{\nabla}(S_a u)\right)_\Gamma\right) + \delta^{-1}\,\widehat{\operatorname{div}}\left(\widehat{\nabla}(S_a u)\right)\right) \text{ in } (H^{1,0}(\Omega_{\delta,\delta a}))^\dagger.$$

Thanks to (3.1.29), we have $\widehat{\operatorname{div}}(\widehat{\nabla}\, S_a u) = u$, which leads to:

$$\operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big) = \mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\left(\widehat{\nabla}(S_a u)\right)_\Gamma\right) + \delta^{-1}\,\mathcal{I}_\delta\, u \text{ in } (H^{1,0}(\Omega_{\delta,\delta a}))^\dagger,$$

an then:

$$\mathcal{I}_\delta\, u = \delta\left(\operatorname{div}_{\mathcal{L}}\Big(\mathcal{I}_\delta\left(\widehat{\nabla}(S_a u)\right)\Big) - \mathcal{I}_\delta\left(\operatorname{div}_\Gamma\left(\left(\widehat{\nabla}(S_a u)\right)_\Gamma\right)\right) \text{ in } (H^{1,0}(\Omega_{\delta,\delta a}))^\dagger.$$

Combining this with (3.1.32) conclude the proof of (3.1.27).

Proof of (3.1.25). Define the two operators A, B for $\mathbf{u} \in H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; (L^2_\#(\hat{Y}_a))^3\right)$ by:

$$A\boldsymbol{u} := \operatorname{div}_{\mathcal{L}}(\tilde{\mathcal{I}}_\delta \mathbf{u}) \quad \text{and} \quad B\boldsymbol{u} := \tilde{\mathcal{I}}_\delta\Big(\operatorname{div}_\Gamma \mathbf{u}_\Gamma + \delta^{-1}\,\widehat{\operatorname{div}}\,\mathbf{u}\Big),$$

and (3.1.25) is equivalent to prove $A = B$. Let us prove $A = B$. Indeed thanks to (3.1.41) we have:

$$(A, B) \in \mathcal{L}\left(H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; (L^2_\#(\hat{Y}_a))^3\right), H^{1,0}(\Omega_{\delta,\delta a})^\dagger\right)^2. \tag{3.1.44}$$

Let $\mathcal{D}_{\#}(\hat{Y}_a)$ be the space of function ψ in $L^2_{\#}(\hat{Y}_a)$ such that $\mathrm{supp}(\psi) \subset \hat{Y}_a$. Thanks to Proposition 2.3.2 and (3.1.27) , we have:

$$\forall u \in C^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathcal{D}_{\#}(\hat{Y}_a)\right), \; Au = Bu. \tag{3.1.45}$$

Moreover we can prove that $C^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathcal{D}_{\#}(\hat{Y}_a)\right)$ is dense into $H^{q+1}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; (L^2_{\#}(\hat{Y}_a))^3\right)$. Therefore (3.1.44) and (3.1.45) conclude the proof of (3.1.25). □

3.2 Stability of the exact problem

Lemma 3.2.1. *There exists $C > 0$ such that for all $\delta > 0$ the following estimate hold:*

$$\|u_\delta\|_{H^1(\Omega_\delta)} \leq C \sup_{\phi \in H^1(\Omega_\delta)} \frac{a_\delta(u_\delta, \phi)}{\|\phi\|_{H^1(\Omega_\delta)}}, \forall u_\delta \in H^1(\Omega_\delta).$$

Proof. It is a proof by contradiction where we are inspired from [37]. Indeed assume that there exists a sequence $(u_\delta)_\delta$ such that:

$$\|u_\delta\|_{H^1(\Omega_\delta)} = 1 \quad \text{and} \quad \lim_{\delta \to 0} \sup_{\phi_\delta \in H^1(\Omega_\delta)} \frac{a_\delta(u_\delta, \phi)}{\|\phi_\delta\|_{H^1(\Omega_\delta)}} = 0. \tag{3.2.46}$$

The difficulty is that our domains Ω_δ depend on δ. That is why we introduce for any sequence of function $(f_\delta)_\delta$ defined on Ω_0, the sequence of function $(\tilde{f}_\delta)$ defined for $(x_\Gamma, \nu) \in \Omega_0$ by

$$\tilde{f}_\delta := f_\delta\big(x_\Gamma, \nu \cdot (1 + \delta/\eta_0) - \delta\big).$$

Using the change of variable formula for integrals yield:

$$\begin{aligned}\|\tilde{u}_\delta\|^2_{H^1(\Omega_0)} =& (1 + \delta/\eta_0)^{-1} \cdot \left(\| \nabla_\Gamma f_\delta\|^2_{L^2(\Omega_\delta)} + \|f_\delta\|^2_{L^2(\Omega_\delta)}\right) + \\ & (1 + \delta/\eta_0) \cdot \|\partial_\nu f_\delta\|^2_{L^2(\Omega_\delta)}.\end{aligned}$$

Therefore we have the following equivalence:

$$\|\tilde{f}_\delta\|_{H^1(\Omega_0)} \underset{\delta \to 0}{\sim} \|f_\delta\|_{H^1(\Omega_\delta)} \quad \text{and} \quad \|\tilde{f}_\delta\|_{L^2(\Omega_0)} \underset{\delta \to 0}{\sim} \|f_\delta\|_{L^2(\Omega_\delta)}. \tag{3.2.47}$$

Thus, thanks to (3.2.46), the sequence $(\tilde{u}_\delta)_{\delta>0}$ is bounded in $H^1(\Omega_0)$. Thus, there exists $\tilde{u}_0 \in H^1(\Omega_0)$. such that up to a sub-sequence the sequence $(\tilde{u}_\delta)_{\delta>0}$ weakly converge to $\tilde{u}_0$ in $H^1(\Omega_0)$.

First, prove that $\tilde{u}_0 = 0$. Let $\phi \in H^1(\Omega_0)$ with $\|\phi\|_{H^1(\Omega_0)} = 1$ and ϕ^δ defined for $(x_\Gamma, \nu) \in \Omega_\delta$ by:

$$\phi^\delta(x_\Gamma, \hat{\nu}) := \phi\left(x_\Gamma, (\nu + \delta)/(1 + \delta/\eta_0)\right).$$

Then from (3.2.47) we get the equivalence $\|\phi^\delta\|_{L^2(\Omega_\delta)} \underset{\delta \to 0}{\sim} 1$ and combining with (3.2.46) yields:

$$a_\delta(u_\delta, \phi^\delta) \underset{\delta \to 0}{\to} 0.$$

Thus we have

$$0 = \lim_{\delta \to 0} \int_{\Omega_\delta} \left(\rho_\delta (\mathcal{C}\, \nabla_{\mathcal{L}}\, u_\delta, \nabla_{\mathcal{L}}\, \phi_\delta) - k^2\, \mathrm{C}\, \mu_\delta u_\delta \overline{\phi_\delta} \right) d\Gamma d\nu + \langle \mathrm{DtN}_{\mathcal{L}}\, u_\delta, \phi \rangle_{\Gamma \times \{\eta_0\}},$$
$$= \lim_{\delta \to 0} \int_{\Omega_0} \left(\tilde{\rho}_\delta (\tilde{\mathcal{C}}\, \nabla_\Gamma\, u_\delta, \nabla_\Gamma\, \phi) + \tilde{\mathrm{C}} \tilde{\rho}_\delta \partial_\nu u_\delta \partial_\nu \overline{\phi} - k^2 \tilde{\mathrm{C}} \tilde{\mu}_\delta \tilde{u}_\delta \overline{\phi} \right) d\Gamma d\nu + \langle \mathrm{DtN}_{\mathcal{L}}\, \tilde{u}_\delta, \phi \rangle_{\Gamma \times \{\eta_0\}},$$

Next using the dominated convergence theorem yields that the sequence of vector:

$$U_\delta := \left(\tilde{\rho}_\delta \tilde{\mathcal{C}}\, \nabla_\Gamma\, \phi ; \tilde{\rho}_\delta \tilde{\mathrm{C}} \partial_\nu \overline{\phi} ; -k^2 \tilde{\mathrm{C}} \tilde{\mu}_\delta \phi \right)$$

strongly converges in $L^2(\Omega_0)$ to:

$$U_0 := \left(\mathcal{C}\, \nabla_\Gamma\, \phi ; \mathrm{C}\, \partial_\nu \phi ; -k^2\, \mathrm{C}\, \phi \right).$$

Thus using that $(\nabla_\Gamma\, \tilde{u}_\delta ; \partial_\nu \tilde{u}_\delta ; \tilde{u}_\delta)$ weakly converges in $L^2(\Omega_0)^3$ to $(\nabla_\Gamma\, \tilde{u}_0 ; \partial_\nu \tilde{u}_0 ; \tilde{u}_\delta)$ we get:

$$\forall \phi \in H^1(\Omega_0), \quad \int_{\Omega_0} \left((\mathcal{C}\, \nabla_{\mathcal{L}}\, \tilde{u}_0, \nabla_{\mathcal{L}}\, \phi) - k^2\, \mathrm{C}\, \tilde{u}_0 \overline{\phi} \right) d\Gamma d\nu + \langle \mathrm{DtN}_{\mathcal{L}}\, \tilde{u}_\delta, \phi \rangle_{\Gamma \times \{\eta_0\}} = 0.$$

Therefore $\tilde{u}_0$ is a solution of the Helmholtz with outgoing condition on $\Gamma \times \{\eta_0\}$ with a zero as right hand-side. Since this problem is well this conclude the proof of $\tilde{u}_0 = 0$.

Secondly, we now prove that $(u_\delta)_{\delta>0}$ strongly converge in $H^1(\Omega_\delta)$ to zero. We remark that the sequence $(\tilde{u}_\delta)_{\delta>0}$ weakly converge to 0 in $H^1(\Omega_0)$. Thus Rellich theorem yields that $(\tilde{u}_\delta)_{\delta>0}$ strongly converges to zero in $L^2(\Omega_0)$. Therefore combining this with (3.2.47) yields that:

$$\lim_{\delta \to 0} \int_{\Omega_\delta} -(k^2 \mu_\delta\, \mathrm{C} + 1) |u_\delta|^2 d\Gamma d\nu = 0. \tag{3.2.48}$$

We define the operator $\mathrm{DtN}^{k=i}_{\mathcal{L}} : H^{\frac{1}{2}}(\Gamma \times \{\eta_0\}) \mapsto H^{-\frac{1}{2}}(\Gamma \times \{\eta_0\})$ for $\tilde{u}_\delta, \tilde{v}_\delta \in H^{\frac{1}{2}}(\Gamma \times \{\eta_0\})$ by:

$$\left\langle \mathrm{DtN}^{k=i}_{\mathcal{L}}\, \tilde{u}_\delta, \tilde{v}_\delta \right\rangle_{\Gamma \times \{\eta_0\}} := \left\langle \mathrm{DtN}^{k=i}\, \tilde{u}_\delta \circ \mathcal{L}, \tilde{u}_\delta \circ \mathcal{L} \right\rangle_{\Sigma_{\eta_0}}$$

where $\mathrm{DtN}^{k=i} : H^{\frac{1}{2}}(\Sigma_0) \mapsto H^{-\frac{1}{2}}(\Sigma_0)$ is the Dirichlet to Neumann map on Σ_0 associated to the wave-number i. Thanks to [60, Theorem 2.6.4], [51, appendix] and [25, Proposition 3.4], we get the compactness of the operator:

$$\mathrm{DtN}_{\mathcal{L}} - \mathrm{DtN}^{k=i}_{\mathcal{L}} : H^{\frac{1}{2}}(\Gamma \times \{\eta_0\}) \mapsto H^{-\frac{1}{2}}(\Gamma \times \{\eta_0\}).$$

Therefore $\lim\limits_{\delta \to 0} \left\langle (\mathrm{DtN}_{\mathcal{L}} - \mathrm{DtN}^{k=i}_{\mathcal{L}}) \tilde{u}_\delta, \tilde{u}_\delta \right\rangle_{\Gamma \times \{\eta_0\}} = 0$. Thus combining this with (3.2.48), (3.2.46) leads to

$$0 = \lim_{\delta \to 0} a_\delta(u_\delta, u_\delta) = \lim_{\delta \to 0} \int_{\Omega_\delta} (\rho_\delta (\mathcal{C}\, \nabla_{\mathcal{L}}\, u_\delta, \nabla_{\mathcal{L}}\, u_\delta) + u_\delta \overline{u_\delta})\, d\Gamma d\nu + \langle \mathrm{DtN}^{k=i}_{\mathcal{L}}\, u_\delta, u_\delta \rangle_{\Gamma \times \{\eta_0\}}.$$

Since the operator $\mathrm{DtN}^{k=i}$ is positive (see[60, Theorem 2.6.4], [51, appendix] and [25, Proposition 3.4]) then $\mathrm{DtN}^{k=i}_{\mathcal{L}}$ is a positive operator in the sense that for all $u_\delta \in H^{\frac{1}{2}}(\Gamma \times \{\eta_0\})$ we have $\langle \mathrm{DtN}^{k=i}_{\mathcal{L}}\, u_\delta, u_\delta \rangle_{\Gamma \times \{\eta_0\}} \geq 0$. Thus we have:

$$\lim_{\delta \to 0} \int_{\Omega_\delta} (\rho_\delta (\mathcal{C}\, \nabla_{\mathcal{L}}\, u_\delta, \nabla_{\mathcal{L}}\, u_\delta) + u_\delta \overline{u_\delta})\, d\Gamma d\nu = 0,$$

and combining this with (4.5.79) yields the final contradiction $\|u_\delta\|_{H^1(\Omega_\delta)} \to 0$. $\square$

3.3 Error decomposition

For what follows in this work n is an arbitrary positive number. Moreover, thanks to Lemma 3.2.1 a sufficient condition is a uniform estimate in $v \in H^1(\Omega_\delta)$ with $\|v\|_{H^1(\Omega_\delta)} = 1$ of $a_\delta(u^\delta - u^n_{\eta,\delta}, v)$. For all $v \in H^1(\Omega_\delta)$, we have the following decomposition of the error:

$$a_\delta(u^\delta - u^n_{\eta,\delta}, v) := \mathcal{D}^r_{\eta,\delta,n} + \mathcal{D}^c_{\eta,\delta,n},$$

where $\mathcal{D}^r_{\eta,\delta,n}$ is so-called "matching error" (it measures the mismatch between the truncated expansions (2.2.7) and (2.2.8):

$$\mathcal{D}^r_{\eta,\delta,n} := \int_{\Omega_\delta} \rho^\delta (u_{n,\delta} - \mathcal{I}_\delta\, \hat{u}_{n,\delta}) \nabla \chi_\eta \nabla \overline{v} - \int_{\Omega_\delta} \rho^\delta \left[\nabla (u_{n,\delta} - \mathcal{I}_\delta\, \hat{u}_{n,\delta}) \nabla \chi_\eta\right] \overline{v}$$

and where $\mathcal{D}^c_{\eta,\delta,n}$ is so-called "consistency error" (it measures how much the truncated expansion (2.2.7) fails to satisfy the original Helmholtz equation):

$$\mathcal{D}^c_{\eta,\delta,n} := a_\delta(\mathcal{I}_\delta\, \hat{u}_{n,\delta}, \chi_\eta v).$$

This error decomposition is exactly the same as in [37]. Thanks to Proposition 2.5.15 and Lemma 2.5.27, we have:

$$\mathcal{T}_k \hat{u}_{n,\delta} \in H^{m_\Gamma + \frac{1}{2} - n}\Big(\Gamma; \mathbb{H}_0\big(\hat{Y}_{\eta/\delta}\big)^\dagger\Big). \tag{3.3.49}$$

We assume that $n \le m_\Gamma - 4$ because this implies $m_\Gamma + \frac{1}{2} - n > 2$ and according to Corollary 3.1.3, (3.3.49) leads to:

$$\mathcal{I}_\delta(\mathcal{T}_k \hat{u}_{n,\delta}) \in \big(H^{1,0}(\Omega_{\delta,\eta})\big)^\dagger.$$

Therefore we now can define the following quantity:

$$\mathcal{D}^{c,0}_{\eta,\delta,n} := \sum_{k=0}^{n} \delta^{-2+k} \langle \mathcal{I}_\delta(\mathcal{T}_k \hat{u}_{n,\delta}), \chi_\eta v \rangle_{\Omega_{\delta,\eta}},$$

which is so-called "first consistency error".

The "second consistency error" is defined by

$$\mathcal{D}^{c,1}_{\eta,\delta,n} := a^\delta(\mathcal{I}_\delta\, \hat{u}_{n,\delta}, \chi_\eta v) - \sum_{k=0}^{n} \delta^{-2+k} \langle \mathcal{I}_\delta(\mathcal{T}_k \hat{u}_{n,\delta}), \chi_\eta v \rangle_{\Omega_{\delta,\eta}},$$

so that the total consistency error has the decomposition $\mathcal{D}^c_{\eta,\delta,n} = \mathcal{D}^{c,0}_{\eta,\delta,n} + \mathcal{D}^{c,1}_{\eta,\delta,n}$.

3.3.1 Estimate of the first consistency error

Here we prove the following result:

Lemma 3.3.1. *The first consistency error satisfies the following estimate:*

$$\mathcal{D}^{c,0}_{\eta,\delta,n} \le C \eta^{n-1} \|v\|_{H^1(\Omega_\delta)}.$$

For all that follow, $C > 0$ is a constant independent of δ and η. The proof of Lemma 3.3.1 is a direct consequence of Proposition 3.3.2 and Proposition 3.3.3. We recall the following useful result(see [67, Lemma 3.10] and [34, equation (3.9.10)])

$$\|\chi_\eta v\|_{H^1(\Omega_{\delta,\eta})} \leq C\eta^{-\frac{1}{2}}\|v\|_{H^1(\Omega_\delta)}. \tag{3.3.50}$$

We define $\langle\cdot,\cdot\rangle_{\Omega_{\delta,\eta}} := \langle\cdot,\cdot\rangle_{H^1(\Omega_{\delta,\eta})^\dagger - H^1(\Omega_{\delta,\eta})}$.

Proposition 3.3.2. *The first consistency error can be rewritten as follow:*

$$\mathcal{D}^{c,0}_{\eta,\delta,n} = \delta^{-2}\sum_{l=n+1}^{2n}\left(\sum_{k=l-n}^{n}\delta^l\langle\mathcal{I}_\delta(\mathcal{T}_k\hat{u}_{l-k}),\chi_\eta v\rangle_{\Omega_{\delta,\eta}}\right), \tag{3.3.51}$$

Proof. By definition we have:

$$\begin{cases} \mathcal{D}^{c,0}_{\eta,\delta,n} = \displaystyle\sum_{k=0}^{n}\delta^{-2+k}\langle\mathcal{I}_\delta(\mathcal{T}_k\hat{u}_{n,\delta}),\chi_\eta v\rangle_{\Omega_{\delta,\eta}}, \\ \qquad\quad = \displaystyle\sum_{(k,l)\in N_1}\delta^{-2+k}\langle\mathcal{I}_\delta(\mathcal{T}_k\delta^l\hat{u}_l),\chi_\eta v\rangle_{\Omega_{\delta,\eta}}, \end{cases}$$

where $N_1 := \{(k,l)\in\mathbb{Z}^2,\ 0\leq k\leq n \quad \text{and} \quad 0\leq l\leq n\}$. Let $\mathcal{N}:\mathbb{Z}^2\mapsto\mathbb{Z}^2$ be defined for $(k,l)\in\mathbb{Z}^2$ by:

$$\mathcal{N}(k,l) := (k,l+k),$$

which is a bijective application. From the following equivalence:

$$\forall(k,l)\in\mathbb{Z}^2,\quad \begin{cases} 0\leq k\leq n \\ 0\leq l\leq n \end{cases} \Longleftrightarrow \left(\begin{cases} 0\leq k\leq l+k \\ 0\leq l+k\leq n \end{cases} \text{ or } \begin{cases} l+k-n\leq k\leq n \\ n+1\leq l+k\leq 2n \end{cases}\right),$$

we get that $N_1 = \mathcal{N}^{-1}(N_2^1\cup N_2^2)$ with $N_2^1\cap N_2^2 = \emptyset$ and:

$$\begin{cases} N_1^1 := \left\{(k,l)\in\mathbb{Z}^2,\ 0\leq k\leq l \quad \text{and} \quad 0\leq l\leq n\right\}, \\ N_2^1 := \left\{(k,l)\in\mathbb{Z}^2,\ l-n\leq k\leq n \quad \text{and} \quad n+1\leq l\leq 2n\right\}. \end{cases}$$

Thus we have:

$$\begin{aligned} \sum_{(k,l)\in N_1}\delta^{-2+k}\langle\mathcal{I}_\delta(\mathcal{T}_k\delta^l\hat{u}_l),\chi_\eta v\rangle_{\Omega_{\delta,\eta}} &= \sum_{(k,l)\in\mathcal{N}(N_1)}\delta^{-2+l}\langle\mathcal{I}_\delta(\mathcal{T}_k\delta^l\hat{u}_{l-k}),\chi_\eta v\rangle_{\Omega_{\delta,\eta}}, \\ &= \sum_{m=1,2}\sum_{(k,l)\in N_2^m}\delta^{-2+l}\langle\mathcal{I}_\delta(\mathcal{T}_k\hat{u}_{l-k}),\chi_\eta v\rangle_{\Omega_{\delta,\eta}}, \end{aligned}$$

where here $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{1,0}(\Omega_\delta)^\dagger - H^{1,0}(\Omega_\delta)}$. Moreover from the relation $\mathcal{T}_0\hat{u}_l + \mathcal{T}_1\hat{u}_{l-1} + \cdots + \mathcal{T}_l\hat{u}_0 = 0,\ \forall l\in\mathbb{N}$, we have:

$$\sum_{(k,l)\in N_2^1}\delta^{-2+l}\langle\mathcal{I}_\delta(\mathcal{T}_k\hat{u}_{l-k}),\chi_\eta v\rangle_{\Omega_{\delta,\eta}} = 0,$$

which leads to:

$$\mathcal{D}^{c,0}_{\eta,\delta,n} = \delta^{-2}\sum_{l=n+1}^{2n}\left(\sum_{k=l-n}^{n}\delta^l\langle\mathcal{I}_\delta(\mathcal{T}_k\hat{u}_{l-k}),\chi_\eta v\rangle_{\Omega_{\delta,\eta}}\right).$$

Thus the proof is finished. □

Proposition 3.3.3. *For all $n+1 \le l \le 2n$ and $l-n \le k \le 2k$ we have the following estimate:*

$$|\langle (\mathcal{T}_k \hat{u}_{l-k})^\delta, \chi_\eta v\rangle_{\Omega_{\delta,\eta}}| \le C\frac{\delta^2}{\delta^l}\eta^{l-2}\|v\|_{H^1(\Omega_\delta)}.$$

Proof. Thanks to Lemma 2.5.27, $\hat{u}_{l-k}$ take the form $\hat{u}_{l-k} = p_{l-k} + R_{l-k}$ with :

$$(p_{l-k}, R_{l-k}) \in H^{m_\Gamma + \frac{1}{2}-(l-k)}\Big(\Gamma; \mathbb{C}_{l-k}[\hat{\nu}]\Big) \times H^{m_\Gamma + \frac{1}{2}-(l-k)}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\right), \tag{3.3.52}$$

and R_{l-k} satisfies the $\mathcal{P}^\infty_{m_\Gamma+\frac{1}{2}-(l-k)}$ property. Therefore thanks to Proposition 2.5.15 there exists:

$$(A_{l,k}, B_{l,k}) \in H^{m_\Gamma+\frac{1}{2}-n-2}\Big(\Gamma; \mathbb{C}_{\max(l-2,0)}[\hat{\nu}]\Big) \times H^{m_\Gamma+\frac{1}{2}-n-2}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big),$$

such that:

$$\mathcal{T}_k \hat{u}_{l-k} = A_{l,k} + B_{l,k}. \tag{3.3.53}$$

Moreover, since we supposed $n \le m_\Gamma - 4$, we have $q_{l,k} := m_\Gamma + \frac{1}{2} - n - 2 > 0$. Therefore thanks to Corollary 3.1.2 we have:

$$\left\|\mathcal{I}_\delta\left(\mathcal{T}_k(\hat{u}_{l-k})\right)\right\|_{H^1(\Omega_{\delta,\eta})^\dagger} \le C\delta^{\frac{3}{2}}\left\|\mathcal{T}_k(\hat{u}_{l-k})\right\|_{H^{q_{l,k}}\left(\Gamma, \mathbb{H}_0\left(\hat{Y}_{\eta/\delta}\right)^\dagger\right)}. \tag{3.3.54}$$

Let us prove:

$$\|A_{l,k}\|_{H^{q_{l,k}}\left(\Gamma, \mathbb{H}_0\left(\hat{Y}_{\eta/\delta}\right)^\dagger\right)} \le C\left(\frac{\eta}{\delta}\right)^{l-\frac{1}{2}}, \tag{3.3.55}$$

and

$$\|B_{l,k}\|_{H^{q_{l,k}}\left(\Gamma, \mathbb{H}_0\left(\hat{Y}_{\eta/\delta}\right)^\dagger\right)} \le C. \tag{3.3.56}$$

Indeed, let us prove (3.3.55). For all $0 \le j \le l-2$, we have thanks to the integral part formula that for all $\phi \in \mathbb{H}_0\big(\hat{Y}_{\eta/\delta}\big)$:

$$\begin{aligned}
\int_{\hat{Y}_{\eta/\delta}} \hat{\nu}^j \phi(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} &= -\int_{\hat{Y}_{\eta/\delta}} \frac{\hat{\nu}^{j+1} - (-\delta)^{j+1}}{j+1} \partial_{\hat{\nu}} \phi(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}, \\
&\le \left\|\frac{\hat{\nu}^{j+1} - (-\delta)^{j+1}}{j+1}\right\|_{L^2(\hat{Y}_{\eta/\delta})} \|\nabla\phi\|_{L^2(\hat{Y}_{\eta/\delta})^3} \le C\left(\frac{\eta}{\delta}\right)^{j+\frac{3}{2}} \|\nabla\phi\|_{L^2(\hat{Y}_{\eta/\delta})^3}.
\end{aligned}$$

Combining this with the definition of the norm of the space $\mathbb{H}_0\big(\hat{Y}_{\eta/\delta}\big)$ given in (3.1.19) yields the following estimate:

$$\forall 0 \le j \le l-2,\ \|\hat{\nu}^j\|_{\mathbb{H}_0\left(\hat{Y}_{\eta/\delta}\right)^\dagger} \le C\left(\frac{\eta}{\delta}\right)^{l-\frac{1}{2}}.$$

Combining this with $A_{l,k} \in H^{q_{l,k}}\Big(\Gamma; \mathbb{C}_{\max(l-2,0)}[\hat{\nu}]\Big)$ conclude the proof of (3.3.55). The estimate (3.3.56) is a direct consequence of $A_{l,k} \in H^{q_{l,k}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big)$ and the definition of the norms of space $\mathbb{H}\big(\hat{Y}_\infty\big)$ and $\mathbb{H}_0\big(\hat{Y}_{\eta/\delta}\big)$.

Combining (3.3.53), (3.3.54), (3.3.55) and (3.3.56) yields:

$$\left\|\mathcal{I}_\delta\left(\mathcal{T}_k(\hat{u}_{l-k})\right)\right\|_{H^1(\Omega_{\delta,\eta})^\dagger} \leq C\delta^{\frac{3}{2}}\left(\frac{\eta}{\delta}\right)^{l-\frac{1}{2}}.$$

Combining this with (3.3.50) yields:

$$|\langle(\mathcal{T}_k\hat{u}_{l-k})^\delta, \chi_\eta v\rangle_{\Omega_{\delta,\eta}}| \leq C\delta^{\frac{3}{2}}\left(\frac{\eta}{\delta}\right)^{l-\frac{1}{2}}\|\chi_\eta v\|_{H^1(\Omega_{\delta,\eta})} \leq \delta^{\frac{3}{2}}\left(\frac{\eta}{\delta}\right)^{l-\frac{1}{2}}\eta^{-\frac{1}{2}}\|v\|_{H^1(\Omega_{\delta,\eta})},$$

which concludes the proof. □

3.3.2 Estimate of the second consistency error

Lemma 3.3.4. *If* $\boxed{n \leq m_\Gamma - 3}$ *then the second consistency error satisfies the following estimate:*

$$\mathcal{D}^{c,1}_{\eta,\delta,n} \leq C\eta^{n+1}\delta^{-1}\|v\|_{H^1(\Omega_\delta)}.$$

We first prove that we can give for u a rigorous meaning to the following equation:

$$\operatorname{div}_{\mathcal{L}}\Big(\rho_\delta\,\mathcal{C}\,\nabla_{\mathcal{L}}\,\mathcal{I}_\delta\,u\Big) + k^2\,\mathrm{C}\,\mu_\delta\,\mathcal{I}_\delta\,u = \mathcal{I}_\delta\left(\delta^{-2}\sum_{n=0}^{\infty}\delta^n\mathcal{T}_n u\right),$$

by giving for all $n \in \mathbb{N}$ an estimate of the following quantity:

$$E_n := \left\|\operatorname{div}_{\mathcal{L}}\Big(\rho_\delta\,\mathcal{C}\,\nabla_{\mathcal{L}}\,\mathcal{I}_\delta\,u\Big) + k^2\,\mathrm{C}\,\mu_\delta\,\mathcal{I}_\delta\,u - \mathcal{I}_\delta\left(\delta^{-2}\sum_{i=0}^{n}\delta^i\mathcal{T}_i u\right)\right\|_{(H^{1,0}(\Omega_{\delta,\eta}))^\dagger}, \qquad (3.3.57)$$

with the following result:

Proposition 3.3.5. *For all* $n > 0$ *there exists* $C_n > 0$ *such that for all* $u \in H^{\frac{1}{2}+3}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_{\eta/\delta}\big)\Big)$ *the following estimate holds:*

$$E_n \leq C\delta^{-\frac{1}{2}}\eta^{n+1}\|u\|_{H^{\frac{1}{2}+2}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)}, \qquad (3.3.58)$$

Proof. Let $n > 0$. Thanks to the Taylor expansion with integral there exists bounded maps: $(\mathcal{R}_n, R_n) : \Omega_\delta \mapsto \mathcal{L}(\mathbb{R}^3) \times \mathbb{R}$ such that for all $(x_\Gamma, \nu) \in \Omega_\delta$ we have:

$$C(x_\Gamma, \nu) = \sum_{i=0}^{n}\mathcal{C}^{(i)}\nu^i + \nu^{n+1}\mathcal{R}_n(x_\Gamma, \nu) \quad \text{and} \quad C(x_\Gamma, \nu) = \sum_{i=0}^{n}c^{(i)}\nu^i + \nu^{n+1}R_n(x_\Gamma, \nu).$$

Hence, we have the following decomposition:

$$\operatorname{div}_{\mathcal{L}}\Big(\rho_\delta\,\mathcal{C}\,\nabla_{\mathcal{L}}\,\mathcal{I}_\delta\,u\Big) + k^2\,\mathrm{C}\,\mu_\delta\,\mathcal{I}_\delta\,u = P(\mathcal{I}_\delta\,u) + R(\mathcal{I}_\delta\,u), \qquad (3.3.59)$$

where we defined P, R for $v \in H^1(\Omega_{\delta,\eta})$ by:

$$\begin{cases} Pv := \displaystyle\sum_{i=0}^{n}\operatorname{div}_{\mathcal{L}}\Big(\rho^\delta\mathcal{C}^{(i)}\rho^\delta\nu^i\,\nabla_{\mathcal{L}}\,v\Big) + k^2\mu^\delta c^{(i)}\nu^i v, \\ Rv := \operatorname{div}_{\mathcal{L}}\Big(\mathcal{R}_n(\nu)\nu^{n+1}\,\nabla_{\mathcal{L}}\,v\Big) + k^2\mu^\delta R_n(\nu)\nu^{n+1}v. \end{cases} \qquad (3.3.60)$$

Since u belongs to $H^{3+\frac{1}{2}}\big(\Gamma, \mathbb{H}\big(\hat{Y}_\infty\big)\big)$, we can prove by using Proposition 3.1.1, Corollary 3.1.3 and Corollary 3.1.2 and doing similar computation as in "Recursive equations for the near field" (see chapter 2) that:

$$P(\mathcal{I}_\delta\, u) = \mathcal{I}_\delta \left(\delta^{-2} \sum_{i=0}^{n} \delta^i \mathcal{T}_i u \right).$$

Combining this with (3.3.59) leads to:

$$E_n = \|R(\mathcal{I}_\delta\, u)\|_{H^1(\Omega_{\delta,\eta})^\dagger}. \tag{3.3.61}$$

Thanks to (3.1.23) we have:

$$\left\|\mathrm{div}_{\mathcal{L}}\Big(\mathcal{R}_n(\nu)\nu^{n+1}\,\nabla_{\mathcal{L}}\,\mathcal{I}_\delta\, u\Big)\right\|_{H^1(\Omega_{\delta,\eta})^\dagger} \leq \left\|\mathcal{R}_n(\nu)\nu^{n+1}\,\nabla_{\mathcal{L}}\,\mathcal{I}_\delta\, u\right\|_{L^2(\Omega_{\delta,\eta})}.$$

Thus using that $\mathcal{R}_n$ is a bounded function and $-\delta \leq \nu \leq \eta$ yields:

$$\left\|\mathrm{div}_{\mathcal{L}}\Big(\mathcal{R}_n(\nu)\nu^{n+1}\,\nabla_{\mathcal{L}}\,\mathcal{I}_\delta\, u\Big)\right\|_{H^1(\Omega_{\delta,\eta})^\dagger} \leq C\eta^{n+1}\|\,\mathcal{I}_\delta\, u\|_{H^1(\Omega_{\delta,\eta})}. \tag{3.3.62}$$

By using similar argument, we prove that:

$$\left\|k^2\mu^\delta R_n(\nu)\nu^{n+1}\right\|_{H^1(\Omega_{\delta,\eta})^\dagger} \leq C\eta^{n+1}\|\,\mathcal{I}_\delta\, u\|_{H^1(\Omega_{\delta,\eta})}. \tag{3.3.63}$$

Moreover since $u \in H^{\frac{1}{2}+2}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma;\mathbb{H}\big(\hat{Y}_{\eta/\delta}\big)\Big)$ we have from Proposition 3.1.3 that:

$$\|\,\mathcal{I}_\delta\, u\|_{H^1(\Omega_{\delta,\eta})} \leq C\delta^{-\frac{1}{2}}\|u\|_{H^{\frac{1}{2}+2}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)}.$$

Combining this with (3.3.62), (3.3.63) and (3.3.60) yields:

$$\|R(\mathcal{I}_\delta\, u)\|_{H^1(\Omega_{\delta,\eta})^\dagger} \leq C\delta^{-\frac{1}{2}}\|u\|_{H^{\frac{1}{2}+2}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)}.$$

Combining this (3.3.61) conclude the proof. □

Proof of Lemma 3.3.4. Since we have $n \leq m_\Gamma - 3$ we can apply Lemma 2.5.27 which yields $\hat{u}_{n,\delta} \in H^{\frac{1}{2}+3}_{0,\Gamma_{\mathrm{M}}}\Big(\Gamma;\mathbb{H}\big(\hat{Y}_{\eta/\delta}\big)\Big)$. Therefore we can apply Proposition 3.3.5 which leads to:

$$\mathcal{D}^{c,1}_{\eta,\delta,n} \leq C_n\eta^{n+1}\delta^{-\frac{1}{2}}\|\hat{u}_{n,\delta}\|_{H^{\frac{1}{2}+2}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)}, \tag{3.3.64}$$

where we recall that:

$$\mathcal{D}^{c,1}_{\eta,\delta,n} := \left| a^\delta(\mathcal{I}_\delta\, \hat{u}_{n,\delta}, \chi_\eta v) - \sum_{k=0}^{n} \delta^{-2+k}\langle \mathcal{I}_\delta(\mathcal{T}_k \hat{u}_{n,\delta}), \chi_\eta v\rangle_{H^{1,0}(\Omega_{\delta,\eta})^\dagger - H^{1,0}(\Omega_{\delta,\eta})} \right|.$$

Let us prove that:

$$\|\hat{u}_{n,\delta}\|_{H^{\frac{1}{2}+2}_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)} \leq C\left(\frac{\eta}{\delta}\right)^{\frac{1}{2}} \tag{3.3.65}$$

We recall from Lemma 2.5.27 the existence of $(R_i)_i \in H^{\frac{1}{2}+2}\big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\big)$ such that:

$$\hat{u}_{n,\delta} = \sum_{i=0}^{n} \delta^i R_i + \sum_{i=0}^{n} \sum_{j=0}^{i} \delta^i p_i^j \hat{\nu}^j, \tag{3.3.66}$$

and for all $0 \leq i \leq n$, $p_i \in H^{2+\frac{1}{2}}(\Gamma; \mathbb{C}_i[\hat{\nu}])$. Thus from Proposition 2.5.18 we get:

$$\|R_i\|_{H^{2+\frac{1}{2}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)} \leq C. \tag{3.3.67}$$

Moreover we have:

$$\left\| \sum_{i=0}^{n} \sum_{j=0}^{i} \delta^i p_i^j \hat{\nu}^j \right\|_{H^{2+\frac{1}{2}}\left(\Gamma;\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)\right)} \leq C \sup_{1\leq i\leq n} \|p_i\|_{H^{2+\frac{1}{2}}(\Gamma;\mathbb{C}_i[\hat{\nu}])} \left(\sum_{i=0}^{n} \sum_{j=0}^{i} \delta^i \|\hat{\nu}^j\|_{\mathbb{H}\left(\hat{Y}_{\eta/\delta}\right)} \right) \leq C \left(\frac{\eta}{\delta}\right)^{\frac{1}{2}}.$$

Combining this with (3.3.64), (3.3.66) and (3.3.67) conclude the proof of (3.3.65). Combining (3.3.65) with (3.3.64) conclude the proof. $\square$

3.3.3 Estimate of matching error

Lemma 3.3.6. *If* $\boxed{n \leq m_\Gamma - 3}$ *then the matching error satisfies the following estimate:*

$$\mathcal{D}^r_{\eta,\delta,n} \leq C \left(\eta^n + \exp\left(-\pi g_{\min} \frac{\eta}{\delta}\right)\right) \|v\|_{H^1(\Omega_\delta)}.$$

Proof. We recall that this quantity is given by:

$$\mathcal{D}^r_{\eta,\delta,n} := \int_{\Omega_\delta} \rho^\delta (u^n_\delta - \mathcal{I}_\delta\, U^n_\delta) \nabla \chi_\eta \nabla \overline{v} - \int_{\Omega_\delta} \rho^\delta \left[\nabla (u^n_\delta - \mathcal{I}_\delta\, U^n_\delta) \nabla \chi_\eta\right] \overline{v}$$

The support of the function $\nabla \chi_\eta$ is given by: $C_\eta := \{(x_\Gamma, \nu),\ \eta < \nu < 2\eta\}$. Thus thanks to the Hölder inequality we get:

$$\mathcal{D}^r_{\eta,\delta,n} \leq \frac{C}{\eta} \left(\|u^n_\delta - \mathcal{I}_\delta\, \hat{u}_{n,\delta}\|_{L^\infty(C_\eta)} \|\nabla v\|_{L^1(C_\eta)} + \|\partial_\nu\, (u^n_\delta - \mathcal{I}_\delta\, \hat{u}_{n,\delta})\|_{L^\infty(C_\eta)} \|v\|_{L^1(C_\eta)} \right). \tag{3.3.68}$$

Let us prove the following estimates

$$\begin{cases} \|u^n_\delta - \mathcal{I}_\delta\, \hat{u}_{n,\delta}\|_{L^\infty(C_\eta)} \leq C\eta \left(\exp\left(-\pi g_{\min} \frac{\eta}{\delta}\right) + \eta^n \right), \\ \|\partial_\nu\, (u^n_\delta - \mathcal{I}_\delta\, \hat{u}_{n,\delta})\|_{L^\infty(C_\eta)} \leq C \left(\exp\left(-\pi g_{\min} \frac{\eta}{\delta}\right) + \eta^n \right). \end{cases} \tag{3.3.69}$$

From $m_\Gamma \geq n+3$ we have for all $0 \leq i \leq n$, $m_\Gamma + 1 - i > \frac{3}{2} + (1 + n - i)$. Thus we can use the Sobolev embedding results:

$$\forall 1 \leq i \leq n,\ H^{m_\Gamma+1-i}(\Omega_0) \subset C^{n+1-i}(\overline{\Omega_0}).$$

Therefore thanks to Lemma 2.5.27 we get for all $1 \leq i \leq n$ that $u_i \in C^{n-i+1}(\overline{\Omega_0})$. Therefore we can use the Taylor expansion with integral reminder, which yields get for all $1 \leq i \leq n$ the existence of $(\mathcal{R}_1^i, \mathcal{R}_2^i) \in C^0(\overline{\Omega_0})^2$ such that for all $(x_\Gamma, \nu) \in \Omega_0$:

$$\begin{cases} u_i(x_\Gamma, \nu) = \displaystyle\sum_{j=0}^{n-i} \frac{\partial_\nu^j u_i(x_\Gamma, 0)}{j!} \nu^j + \nu^{n-i+1} \mathcal{R}_1^i(x_\Gamma, \nu), \\ \partial_\nu u_i(x_\Gamma, \nu) = \displaystyle\sum_{j=0}^{n-i-1} \frac{\partial_\nu^{j+1} u_i(x_\Gamma, 0)}{j!} \nu^j + \nu^{n-i} \partial_\nu \mathcal{R}_2^i(x_\Gamma, \nu). \end{cases}$$

Thanks to (2.3.35) we have:

$$(\dagger) \begin{cases} u_\delta^n(x_\Gamma, \nu) = \displaystyle\sum_{i=0}^{n} \sum_{j=0}^{n-i} \delta^i \frac{\partial_\nu^j u_i(x_\Gamma, 0)}{j!} \nu^j + \sum_{i=0}^{n} \delta^i \nu^{n-i+1} \mathcal{R}_1^i \\ \qquad = \displaystyle\sum_{i=0}^{n} \sum_{j=0}^{n-i} \delta^j p_j^{i+j}(x_\Gamma) \nu^i + \sum_{i=0}^{n} \delta^i \nu^{n-i+1} \mathcal{R}_1^i, \\ \partial_\nu u_\delta^n(x_\Gamma, \nu) = \displaystyle\sum_{i=0}^{n} \sum_{j=0}^{n-i-1} \delta^i \frac{\partial_\nu^{j+1} u_i(x_\Gamma, 0)}{j!} \nu^{j-1} + \sum_{i=0}^{n} \delta^i \nu^{n-i} \mathcal{R}_2^i \\ \qquad = \displaystyle\sum_{i=0}^{n} \sum_{j=1}^{n-i} \delta^j j p_j^{i+j}(x_\Gamma) \nu^{j-1} + \sum_{i=0}^{n} \delta^i \nu^{n-i} \mathcal{R}_2^i. \end{cases}$$

where for all $0 \leq n' \leq n$, $(p_k^{n'}(x_\Gamma)_k$ is the coefficient of the polynomial $p_n(x_\Gamma)$ appearing in Lemma 2.5.27. Moreover thanks to Lemma 2.5.27 for all $1 \leq i \leq n$ there exists $R_i \in H^{m_\Gamma + \frac{1}{2} - i}(\Gamma; \mathbb{H}(\hat{Y}_\infty))$ satisfying the $\mathcal{P}^\infty_{m_\Gamma + \frac{1}{2} - i}$ property such that for all:

$$\forall (x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \in \hat{Y}_\infty, \ \hat{u}_i(x_\Gamma; \hat{x}, \hat{\nu}) = p_i(x_\Gamma; \hat{\nu}) + R_i(x_\Gamma; \hat{x}, \hat{\nu}).$$

Therefore:

$$(\dagger\dagger) \begin{cases} \mathcal{I}_\delta \, \hat{u}_{n,\delta} = \displaystyle\sum_{i=1}^{n} \delta^i \, \mathcal{I}_\delta \, R_i + \sum_{k=0}^{n} \sum_{l=0}^{k} \delta^{k-l} p_l^k \nu^l = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \delta^j p_j^{i+j} \nu^j + \sum_{i=1}^{n} \delta^i \, \mathcal{I}_\delta \, R_i, \\ \partial_\nu(\mathcal{I}_\delta \, \hat{u}_{n,\delta}) = \displaystyle\sum_{i=1}^{n} \delta^{i-1} \, \mathcal{I}_\delta(\partial_{\hat{\nu}} R_i) + \sum_{k=0}^{n} \sum_{l=1}^{k} \delta^{k-l} p_l^k l \nu^{l-1} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \delta^j j p_j^{i+j} \nu^{j-1} + \sum_{i=1}^{n} \delta^{i-1} \, \mathcal{I}_\delta(\partial_{\hat{\nu}} R_i). \end{cases}$$

Doing the difference between $(\dagger)$ and $(\dagger\dagger)$ yields:

$$\begin{cases} u_\delta^n - \mathcal{I}_\delta \, \hat{u}_{n,\delta} = \displaystyle\sum_{i=0}^{n} \delta^i \nu^{n-i+1} \mathcal{R}_1^i - \sum_{i=1}^{n} \delta^i \, \mathcal{I}_\delta \, R_i \\ \partial_\nu \left(u_\delta^n - \mathcal{I}_\delta \, \hat{u}_{n,\delta} \right) = \displaystyle\sum_{i=0}^{n} \delta^i \nu^{n-i} \mathcal{R}_2^i - \sum_{i=1}^{n} \delta^{i-1} \, \mathcal{I}_\delta(\partial_{\hat{\nu}} R_i) \end{cases} \tag{3.3.70}$$

On C_η we have $\nu \leq 2\eta$ and we recall that $\delta \leq \eta$. Therefore we have:

$$\left\| \sum_{i=1}^{n} \delta^i \nu^{n-i+1} \mathcal{R}_1^i \right\|_{L^\infty(C_\eta)} \leq C\eta^{n+1} \quad \text{and} \quad \left\| \sum_{i=0}^{n} \delta^i \nu^{n-i} \mathcal{R}_2^i \right\|_{L^\infty(C_\eta)} \leq C\eta^n. \tag{3.3.71}$$

Let us prove that for all $1 \leq i \leq n$ we have:

$$\|(R_i, \partial_{\hat{\nu}} R_i)\|_{L^\infty(C_\eta)} \leq C \exp\Big(-\pi g_{\min} \frac{\eta}{\delta}\Big). \tag{3.3.72}$$

Indeed let $1 \leq i \leq n$. We recall that that R_i satisfies $\mathcal{P}^\infty_{m_\Gamma - n + \frac{1}{2}}$ property. Therefore according to Proposition 2.5.18 we get that:

$$\exp(\pi g_{\min} \hat{\nu})(R_i, \partial_{\hat{\nu}} R_i) \in H^{m_\Gamma - n + \frac{1}{2}}_{0,\Gamma_{\mathrm{M}}} \left(\Gamma; C_0\big([0,1] \times [1,\infty[\big)\right).$$

Moreover, we have $m_\Gamma \geq n+3$. Therefore, we have thanks to Sobolev injection results

$$H^{m_\Gamma - n + \frac{1}{2}}_{0,\Gamma_{\mathrm{M}}} \left(\Gamma; C_0\big([0,1]^2 \times [1,\infty[\big)\right) \subset L^\infty(\Gamma \times \hat{Y}_\infty).$$

Therefore for all $1 \leq i \leq n$ and $(x_\Gamma, \nu) \in \Gamma \times]\eta, 2\eta[$ we have:

$$\begin{aligned}
\exp\left(\pi g_{\min} \frac{\eta}{\delta}\right) \left| \mathcal{I}_\delta \left((R_i, \partial_{\hat{\nu}} R_i)\right)(x_\Gamma, \nu)\right| &\leq \exp\left(\pi g_{\min} \frac{\nu}{\delta}\right) \left| \mathcal{I}_\delta \left((R_i, \partial_{\hat{\nu}} R_i)\right)(x_\Gamma, \nu)\right|, \\
&\leq \left| \exp\left(\pi g_{\min} \frac{\nu}{\delta}\right) (R_i, \partial_{\hat{\nu}} R_i) \left(x_\Gamma, \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\nu}{\delta}\right)\right|, \\
&\leq \|\exp(\pi g_{\min} \hat{\nu})(R_i, \partial_{\hat{\nu}} R_i)\|_{L^\infty(\Gamma \times \hat{Y}_\infty)},
\end{aligned}$$

which conclude the proof of (3.3.72).

Combining (3.3.72) and (3.3.71) with (3.3.70) conclude the proof of (3.3.69).

Combining $\|\nabla v\|_{L^1(C_\eta)} \leq C \|v\|_{H^1(\Gamma \times]-\delta, \eta_0[)}$ with (3.3.69) yields:

$$\frac{C}{\eta} \left(\|u^n_\delta - \mathcal{I}_\delta\, \hat{u}_{n,\delta}\|_{L^\infty(C_\eta)} \|\nabla v\|_{L^1(C_\eta)}\right) \leq \left(\exp\left(-\pi g_{\min} \frac{\eta}{\delta}\right) + \eta^n\right) \|v\|_{H^1(\Gamma \times]-\delta, \eta_0[)}. \tag{3.3.73}$$

Moreover thanks to Cauchy-Schwartz we get $\|v\|_{L^1(C_\eta)} \leq C\eta^{\frac{1}{2}} \|v\|_{L^2(C_\eta)}$. Combining this with the classical estimate (3.3.50) that we recall here:

$$\|v\|_{L^2(C_\eta)} \leq C\eta^{\frac{1}{2}} \|v\|_{H^1(C_\eta)},$$

yields the estimate $\|v\|_{L^1(C_\eta)} \leq C\eta \|v\|_{H^1(\Gamma \times]-\delta, \eta_0[)}$. Combining this with (3.3.69) yields:

$$\frac{C}{\eta} \left(\|\partial_\nu \left(u^n_\delta - \mathcal{I}_\delta\, \hat{u}_{n,\delta}\right)\|_{L^\infty(C_\eta)} \|v\|_{L^1(C_\eta)}\right) \leq \left(\exp\left(-\pi g_{\min} \frac{\eta}{\delta}\right) + \eta^n\right) \|v\|_{H^1(\Gamma \times]-\delta, \eta_0[)}. \tag{3.3.74}$$

The proof is finished because combining (3.3.73) and (3.3.74) with (3.3.68) yields the desired estimate.

3.4 Justification theorem

Theorem 3.4.1. *If* $\boxed{n \leq m_\Gamma - 6}$ *then for all* $c > 0$ *there exists* $C > 0$ *such that the following estimate holds:*

$$\|u_\delta - u_{n,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)} \leq C\delta^{n+1}.$$

Proof. Since we have

$$\lim_{\delta\to 0} \delta^{\frac{n+1}{n+2}-1} = \infty \quad \text{and} \quad \lim_{\delta\to 0} \delta^{\frac{n+1}{n+2}} = 0,$$

the following choice:

$$\eta := \delta^{\frac{n+1}{n+2}},$$

is suitable in the sense that it satisfies (2.2.6). Since we have $n+3 \leq m_\Gamma - 3$ we can apply Lemma 3.3.1, Lemma 3.3.4, Lemma 3.3.6 and Lemma 3.2.1 with $n = n+3$ which leads to

$$\|u_\delta - u^{n+3}_{\eta,\delta}\|_{H^1(\Omega_\delta)} \leq C\left(\delta^{n+1} + \delta^{\frac{(n+4)(n+1)}{(n+2)}-1} + \exp\left(-\pi g_{\min}\delta^{-\frac{1}{n+2}}\right)\right). \tag{3.4.75}$$

Moreover we can prove that

$$\delta^{\frac{(n+4)(n+1)}{(n+2)}-1} \leq \delta^{n+1}. \tag{3.4.76}$$

Moreover since the exponential function is strongly decreasing a infinity we have:

$$\exp\left(-\pi\delta^{-\frac{1}{n+2}}\right) \leq C\delta^{n+1}. \tag{3.4.77}$$

Therefore combining (3.4.75), (3.4.76) and (3.4.77) yields:

$$\|u_\delta - u^{n+3}_{\eta,\delta}\|_{H^1(\Gamma\times]-\delta,\eta_0[)} \leq C\delta^{n+1}. \tag{3.4.78}$$

Since η tend to zero we can assume that $c \geq 2\eta$ which leads to:

$$\|u_\delta - u_{n+3,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)} = \|u_\delta - u^{n+3}_{\eta,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)}.$$

Thus we have thanks to triangular inequality:

$$\|u_\delta - u_{n,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)} \leq \|u_\delta - u^{n+3}_{\eta,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)} + \delta^{n+1}\sum_{i=n+1}^{n+1} \|u_j\|_{H^1(\Gamma\times]c,\eta_0[)}.$$

Combining this last estimate with (3.4.78) yields the desired estimate and then the proof is finished. ∎

Part II

Construction and analysis of approximate boundary conditions and numerical results

Table of Contents

Chapter 4

Construction and analysis of approximate boundary conditions

In the previous part we succeed to construct for all $n \in \mathbb{N}$ an approximation of the form $u^0 + \delta u^1 + \cdots + \delta^n u^n$. The procedure of computation of each u^i stated in the previous part is not yet useable. First we will the recall result of the first part. Secondly we will simplify the expressions of the terms u^0, u^1 and u^2. Afterward we will deduce from these simplified expression a second approximated model using impedance boundary condition. Next we will prove that the models with impedance boundary condition are well posed and stable with respect to the small parameter δ. Finally we will deduce error estimates.

To compute the coefficients appearing in the first order impedance boundary condition, one needs to compute for all x_Γ, two functions $w_1(x_\Gamma;\cdot)$ and $w_2(x_\Gamma;\cdot)$ which are solution of partial differential equation on a semi infinite bands (See Figure 4.2). The curvature of our surface Γ will appears in the coefficient in the second impedance boundary condition.

The expression of the second order impedance condition is not very useable. However we found an assumption of symmetric on the reference functions associated to our physical coefficients such that the expressions of the second order impedance boundary conditions are simplified. In this case we do not need to compute other solution of partial differential equation than $w_1(x_\Gamma;\cdot)$ and $w_2(x_\Gamma;\cdot)$.

We can refer the reader to [9, 10], [40], [14], [13] ,[11]and [63] for use of impedance boundary conditions to approximate for an example homogeneous thin coat. Moreover we can also refer the reader for the study inverse problem with impedance boundary conditions to [25, 50, 26, 21, 22].

4.1 Results of Part I

Let us recall the geometry of our problem. The obstacle O is a bounded domain of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus O$ is connected with boundary Γ with $m_\Gamma + 1$ regularity. The "thin coating of width δ" is the following subset C^δ of O:

$$C^\delta := \{x \in O,\ \operatorname{dist}(x, \Gamma) < \delta\},$$

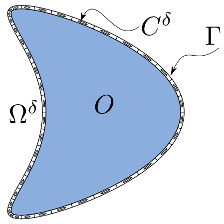


Figure 4.1: Illustration of the geometry

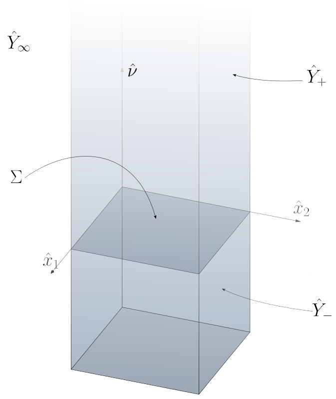


Figure 4.2: Illustration the of infinite strip

where $\delta > 0$ is a small parameter. Here the quantity $\operatorname{dist}(x, \Gamma)$ is the distance of x from the surface Γ defined by

$$\operatorname{dist}(x, \Gamma) := \inf_{x_\Gamma \in \Gamma} |x - x_\Gamma|,$$

and $|.|$ is the classical euclidean norm of $\mathbb{R}^3$. We need to introduce the complementary of O in $\mathbb{R}^3$ $\Omega := \mathbb{R}^3 \backslash \overline{O}$ and $\Omega^\delta := \overline{\Omega} \cup C^\delta$. We refer the reader to Figure 4.1 for an illustration in 2D. The function u^δ is defined as the unique solution of: Find $u^\delta \in H^1_{\text{loc}}(\Omega^\delta)$ such that:

$$\begin{cases} \operatorname{div}\big(\rho^\delta \, \nabla u^\delta\big) + k^2 \mu^\delta \, u^\delta = f, & \text{in } \Omega^\delta, \\ \partial_{n^\delta} u_\delta = 0 & \text{on } \partial\Omega^\delta, \end{cases} \tag{4.1.1}$$

and u_δ satisfies the Sommerfeld radiation condition:

$$\lim_{R \to \infty} \int_{|x|=R} |\partial_r u^\delta - iku^\delta|^2 = 0.$$

Here n^δ and n are respectively the unit outward normal to $\partial\Omega^\delta$ and Ω, $k \in \mathbb{R}$ is the wave-number and f denotes a given source term. Moreover ρ^δ, μ^δ denote the acoustical characteristics of the medium supposed to be equal to 1 in Ω and $\delta-$ periodic in the thin coating C^δ. We do not need for this part to recall the definition of the ψ_Γ $-\delta-$periodicity.

In chapter 3, we succeed to construct an approximation of our exact solution through function $\hat{u} : (x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty \mapsto \hat{u}(x_\Gamma; \hat{x}, \hat{\nu})$ where we introduced the infinite strip (see

Figure 4.2):

$$\hat{Y}_\infty :=]0,1[^2 \times]-1,\infty[.$$

Functions defined on $\Gamma \times \hat{Y}_\infty$ are seen as functions defined on Γ that values are function $x_\Gamma \in \Gamma \mapsto \hat{u}(x_\Gamma;\cdot)$.

In what follow, we recall useful differential operators that apply of functions defined on $\Gamma \times \hat{Y}_\infty$.

First we define the operators $\widehat{\nabla}$ and $\widehat{\mathrm{div}}$ that only concern the variables $\hat{x}$ and $\hat{\nu}$. (That means that x_Γ plays essentially the role of a parameter).

The operator $\widehat{\nabla}$ is defined for $u : \Gamma \mapsto H^1_{\mathrm{loc}}(\hat{Y}_\infty)$ by the function $\widehat{\nabla}\, u : \Gamma \mapsto L^2_{\mathrm{loc}}(\hat{Y}_\infty)$ defined as follow:

- For $x_\Gamma \in \Gamma_{\mathrm{M}}$ and $(\hat{x},\hat{\nu}) \in \hat{Y}_\infty$:

$$\widehat{\nabla}\, u(x_\Gamma;\hat{x},\hat{\nu}) := \underbrace{\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger \nabla_{\hat{x}} u(x_\Gamma;\hat{x},\hat{\nu})}_{\in T_{x_\Gamma}\Gamma} + \underbrace{n(x_\Gamma)\partial_{\hat{\nu}} u(x_\Gamma;\hat{x},\hat{\nu})}_{\in T_{x_\Gamma}\Gamma^\perp}, \tag{4.1.2}$$

  where we recall that $T_{x_\Gamma}\Gamma := n(x_\Gamma)^\perp$, $\mathrm{D}\,\psi_\Gamma(x_\Gamma) : T_{x_\Gamma}\Gamma \mapsto \mathbb{R}^2$ and $\mathrm{D}^\dagger\,\psi_\Gamma(x_\Gamma) : \mathbb{R}^2 \mapsto T_{x_\Gamma}\Gamma$.

- For $x_\Gamma \notin \Gamma_{\mathrm{M}}$:

$$\widehat{\nabla}\, u(x_\Gamma;\hat{x},\hat{\nu}) := n(x_\Gamma)\partial_{\hat{\nu}} u(x_\Gamma;\hat{x},\hat{\nu}).$$

Then, we recall that the operator $\widehat{\mathrm{div}}$ is defined for $u : \Gamma \mapsto L^2_{\mathrm{loc}}$ and $x_\Gamma \in \Gamma$ by the element of $\mathcal{D}\left(\overline{\hat{Y}_\infty}\right)^\dagger$ defined for $\psi \in \mathcal{D}\left(\overline{\hat{Y}_\infty}\right)$ by:

$$\langle \widehat{\mathrm{div}}\, u(x_\Gamma;\cdot), v\rangle_{\mathcal{D}\left(\overline{\hat{Y}_\infty}\right)^\dagger - \mathcal{D}\left(\overline{\hat{Y}_\infty}\right)} := -\int_{\hat{Y}_\infty} u(x_\Gamma;\cdot)\cdot \widehat{\nabla}\, v d\hat{x} d\hat{\nu}. \tag{4.1.3}$$

We define a family of differential operators with respect to all variable $(x_\Gamma,\hat{x},\hat{\nu})$ (Combination of 3D differential operators in $(\hat{x},\hat{\nu})$: $\widehat{\nabla}$ and $\widehat{\mathrm{div}}$ that are referred with a hat with tangential differential operators in the variable x_Γ : div_Γ and ∇_Γ that are referred with a subscript Γ). We recall that the sequence of operator $(\mathcal{T}_k)_{k\in\mathbb{N}}$ is defined for $k \in \mathbb{N}$ by:

$$\mathcal{T}_k := \sum_{j=0}^{2} \mathcal{T}_{k,j}, \tag{4.1.4}$$

where $(\mathcal{T}_{k,j})_{0\le j\le 2}$ are defined for $\hat{u} : (x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma \times \hat{Y}_\infty \mapsto \hat{u}(x_\Gamma;\hat{x},\hat{\nu})$ and $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma \times \hat{Y}_\infty$ by:

$$\begin{cases} \mathcal{T}_{k,0}\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) := \mathrm{div}_\Gamma\left(\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{C}^{(k-2)}(x_\Gamma)\hat{\nu}^{k-2}\,\nabla_\Gamma\,\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\right), \\ \qquad\qquad\qquad\quad +k^2\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\hat{\nu}^{k-2}c^{(k-2)}(x_\Gamma)\hat{u}(x_\Gamma;\hat{x},\hat{\nu}), \\ \mathcal{T}_{k,1}\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) := \mathrm{div}_\Gamma\left(\hat{\nu}^{k-1}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{C}^{(k-1)}(x_\Gamma)\,\widehat{\nabla}\,\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\right), \\ \qquad\qquad\qquad\quad +\widehat{\mathrm{div}}\left(\hat{\nu}^{k-1}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{C}^{(k-1)}(x_\Gamma)\,\nabla_\Gamma\,\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\right), \\ \mathcal{T}_{k,2}\hat{u}(x_\Gamma;\hat{x},\hat{\nu}) := \widehat{\mathrm{div}}\left(\hat{\nu}^k\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{C}^{(k)}(x_\Gamma)\,\widehat{\nabla}\,\hat{u}(x_\Gamma;\hat{x},\hat{\nu})\right). \end{cases}$$

Here, for all $k \in \mathbb{Z}$, $\big(\mathcal{C}^{(k)}, c^{(k)}\big)$ are elements of $C^{m_\Gamma+1-k}\big(\Gamma;\mathcal{L}(\mathbb{R}^3)\big) \times C^{m_\Gamma+1-k}\big(\Gamma\big)$ defined for $x_\Gamma \in \Gamma$ by:

$$\mathcal{C}^{(k)}(x_\Gamma) := \frac{1}{k!}\partial_\nu^k \mathcal{C}(x_\Gamma, 0) \quad \text{if} \quad k \geq 0 \quad \text{else} \quad \mathcal{C}^{(k)}(x_\Gamma) := 0,$$

and $c^{(k)}(x_\Gamma) := \big(\mathcal{C}^{(k)}(x_\Gamma, 0)\cdot n(x_\Gamma)\big)\cdot n(x_\Gamma)$. The linear operator $\mathcal{C} : \Gamma \times \mathbb{R} \mapsto \mathcal{L}(\mathbb{R}^3)$ is defined for $(x_\Gamma, \nu) \in \Gamma \times \mathbb{R}$ by the only linear operator such that for all $v_\Gamma \in T_{x_\Gamma}\Gamma$

$$\mathcal{C}(x_\Gamma,\nu)\cdot v_\Gamma := \mathrm{C}(x_\Gamma,\nu)\cdot\big(\mathbb{I}+\nu R(x_\Gamma)\big)^{-2}\cdot v_\Gamma \quad \text{and} \quad \mathcal{C}(x_\Gamma,\nu)\cdot n(x_\Gamma) := \mathrm{C}(x_\Gamma)\cdot n(x_\Gamma),$$

and $R(x_\Gamma)$ is the tensor curvature defined as follow: We extend the unit outward normal application $n : \Gamma \mapsto \mathbb{R}^3$ for x near from the boundary Γ which takes the form $x = x_\Gamma + \nu n(x_\Gamma)$ by $n(x) := n(x_\Gamma)$. Then the tensor curvature R is defined for x_Γ by:

$$R(x_\Gamma) := \mathrm{D}\, n(x_\Gamma),$$

where D is the classical differential 3×3 matrix. We recall that for all $x_\Gamma \in \Gamma$ we have $\mathrm{Im}\,(R(x_\Gamma)) \subset T_{x_\Gamma}\Gamma$ and $R(x_\Gamma) : T_{x_\Gamma}\Gamma \mapsto T_{x_\Gamma}\Gamma$ is a symmetric tensor. In particular we recall that for all $x_\Gamma \in \Gamma$:

$$\begin{cases} \mathcal{C}^{(0)}(x_\Gamma) = \mathbb{I}, & c^{(0)}(x_\Gamma) = 1, \\ \mathcal{C}^{(1)}(x_\Gamma) = 2\big(H(x_\Gamma) - R(x_\Gamma)\big), & c^{(1)}(x_\Gamma) = 2H(x_\Gamma) \end{cases}, \tag{4.1.5}$$

where $H(x_\Gamma) := \dfrac{\mathrm{tr}(R(x_\Gamma))}{2}$.

We recall the following definition:

$$\mathbb{H}\big(\hat{Y}_\infty\big) := \left\{ u \in H^1_{\mathrm{loc}}(\hat{Y}_\infty),\ \|u\|^2_{\mathbb{H}\left(\hat{Y}_\infty\right)} := \int_{\hat{Y}_\infty} |\nabla u|^2 d\hat{x}d\hat{\nu} + \left| \int_\Sigma u d\hat{x} \right|^2 < \infty \text{ and } u \text{ is one periodic in } \hat{x} \right\},$$

where $\Sigma :=]0,1[^2\times\{0\}$. We recall that $1 \in \mathbb{H}\big(\hat{Y}_\infty\big)$ and then μ is defined for $q \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \oplus \mathbb{C}[\hat{\nu}]$ by:

$$\mu\Big(q\Big) := \langle q_1, 1\rangle_{\hat{Y}_\infty} + \int_{-1}^0 q_2 \mathrm{d}\hat{\nu}, \tag{4.1.6}$$

with $q = q_1 + q_2$ for some $(q_1, q_2) \in \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \times \mathbb{C}[\hat{\nu}]$ (We recall that Proposition 2.5.9 (cf Chapter 2) states that this decomposition is unique). In (4.1.6) and hereafter $\langle\cdot,\cdot\rangle_{\hat{Y}_\infty}$ is the dual bracket between $\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ and $\mathbb{H}\big(\hat{Y}_\infty\big)$. We recall that we constructed an operator $\mathcal{T}_0^{-1}$ defined for map $f : \Gamma \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \oplus \mathbb{C}[\hat{\nu}] \mapsto \mathbb{H}\big(\hat{Y}_\infty\big) \times \mathbb{C}[\hat{\nu}]$ such that for all $f : \Gamma \mapsto \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \times \mathbb{C}[\hat{\nu}]$ we have $\mathcal{T}_0^{-1} f : \Gamma \mapsto \mathbb{H}\big(\hat{Y}_\infty\big) \oplus \mathbb{C}[\hat{\nu}] \mapsto \mathbb{H}\big(\hat{Y}_\infty\big) \times \mathbb{C}[\hat{\nu}]$ and $\mathcal{T}_0\mathcal{T}_0^{-1} f = f$.

Finally, thanks these last reminders, we can recall that the sequences $(u_n)_{n\in\mathbb{N}}$ and $(\hat{u}_n)_{n\in\mathbb{N}}$ are defined by induction. Now we present the process of construction of u_n and $\hat{u}_n$ from the knowledge of $(\hat{u}_k, u_k)$ for $k = 0, 1, \ldots, n-1$.

4.1.1 Construction of the far field u_n

The far fields $u_n : \Omega_0 \mapsto \mathbb{C}$ is defined as the unique solution of: Find $u_n \in H^1\big(\Omega_0\big)$ such that for all $v \in \Gamma \times H^1(\Omega_0)$ we have:

$$a_0(u_n, v) = \begin{cases} \langle f_{\Sigma_{\eta_0}}, v\rangle_{\Gamma\times\{\eta_0\}} \text{ if } n = 0, \\ -\langle l_n, v\rangle_{\Gamma\times\{0\}} \text{ if } n \neq 0, \end{cases} \tag{4.1.7}$$

where we recall that:

- $\Omega_0 := \Gamma\times]0, \eta_0[$, $\Gamma \times \{0\} := \Gamma \times \{0\}$, $\Gamma \times \{\eta_0\} := \Gamma \times \{\eta_0\}$. (see Figure 4.3)
- For all $0 \leq n \leq m_\Gamma$:
$$u_n \in H^{m_\Gamma - n+1}(\Omega_0). \tag{4.1.8}$$
- $\langle \cdot, \cdot\rangle_{\Gamma\times\{0\}} := \langle \cdot, \cdot\rangle_{H^{-\frac{1}{2}}(\Gamma\times\{0\})-H^{\frac{1}{2}}(\Gamma\times\{0\})}$ and $\langle \cdot, \cdot\rangle_{\Gamma\times\{\eta_0\}} := \langle \cdot, \cdot\rangle_{H^{-\frac{1}{2}}(\Gamma\times\{0\})-H^{\frac{1}{2}}(\Gamma\times\{\eta_0\})}$.
- The function $l_n : \Gamma \mapsto \mathbb{C}$ is defined for $x_\Gamma \in \Gamma$ by:
$$l_n(x_\Gamma) := \sum_{i=1}^{n+1} \sum_{j=1}^{n+1-i} \mu\Big(\big(\mathcal{T}_i \mathcal{T}_0^{-1} \mathcal{T}_j \hat{u}_{n+1-i-j}\big)(x_\Gamma; \cdot)\Big) - \sum_{i=1}^{n+1} \mu\Big(\big(\mathcal{T}_i u_{n+1-j}\big)(x_\Gamma; \cdot)\Big). \tag{4.1.9}$$
To give a sense of this last definition, we emphasize that for all $x_\Gamma \in \Gamma$, $1 \leq i \leq n+1$ and $1 \leq j \leq n+1-i$ that $\big(\mathcal{T}_i \mathcal{T}_0^{-1} \mathcal{T}_j \hat{u}_{n+1-i-j}\big)(x_\Gamma; \cdot)$ and $\big(\mathcal{T}_i u_{n+1-j}\big)(x_\Gamma; \cdot)$ are element of $\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger \oplus \mathbb{C}[\hat{\nu}]$.
- The sesquilinear form $a_0 : H^1\big(\Omega_0\big)^2 \mapsto \mathbb{C}$ is defined for $(u, v) \in H^1\big(\Omega_0\big)^2$ by:
$$a_0(u, v) := \int_{\Omega_0} \big(\nabla_{\mathcal{L}}\, u \cdot \nabla_{\mathcal{L}}\, v - k^2 u\overline{v}\big)\, d\Gamma d\nu + \langle \mathrm{DtN}_{\mathcal{L}}\, u, v\rangle_{\Gamma\times\{\eta_0\}},$$
where:
  - For all $u \in H^1\big(\Omega_0\big)$, $\nabla_{\mathcal{L}}\, u := \nabla_\Gamma\, u + n\partial_{\hat{\nu}} u$
  - The inverse of the map $\mathcal{L}$ is defined by $\mathcal{L}^{-1} : \Omega_\delta \mapsto C^{\eta_0}_{\text{ext}}$ is given for $(x_\Gamma, \hat{\nu})$ by
  $$\mathcal{L}^{-1}(x_\Gamma, \hat{\nu}) := x_\Gamma + n(x_\Gamma)\nu,$$
  where $C^{\eta_0}_{\text{ext}} := \{x \in \Omega, \mathrm{dist}(x, \Gamma) < \eta_0\}$ and this last application is a C^{m_Γ} diffeomorphism. (see Figure 4.3)
  - $\mathrm{DtN}_{\mathcal{L}} : H^{\frac{1}{2}}\left(\Gamma \times \{\eta_0\}\right) \mapsto H^{-\frac{1}{2}}\left(\Gamma \times \{\eta_0\}\right)$ is defined for $(u, v) \in H^{\frac{1}{2}}\left(\Gamma \times \{\eta_0\}\right)^2$ by:
  $$\langle \mathrm{DtN}\, \tilde{u}_\delta \circ \mathcal{L}, \tilde{u}_\delta \circ \mathcal{L}\rangle_{\Sigma_{\eta_0}},$$
  where the Dirichlet to Neumann map on $\Sigma_{\eta_0} := \big\{x \in \Omega^\delta, \mathrm{dist}(x, \Gamma) = \eta_0\big\}$:
  $$\mathrm{DtN} : H^{\frac{1}{2}}\left(\Sigma_{\eta_0}\right) \mapsto H^{-\frac{1}{2}}\left(\Sigma_{\eta_0}\right)$$

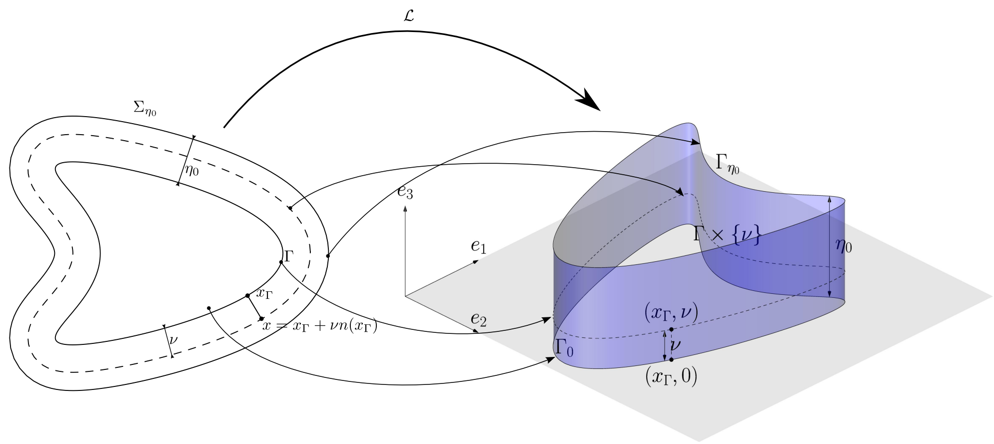


Figure 4.3: The map $\mathcal{L}$

is defined for $g \in H^{\frac{1}{2}}(\Sigma_{\eta_0})$ by $\mathrm{DtN}\, g := \partial_\nu u_g$ and u_g is the unique solution of: Find $u_g \in H^1_{\mathrm{loc}}(\Omega \setminus C_{\delta,\eta_0})$ such that :

$$\begin{cases} \Delta u_g + k^2 u_g = 0, & \text{in } \Omega \setminus C^{\eta_0}_{\mathrm{ext}}, \\ u_g = g & \text{on } \Sigma_{\eta_0} \end{cases}$$

and u_g satisfies the Sommerfeld radiation condition.

- $f_{\Sigma_{\eta_0}} \in H^{-\frac{1}{2}}(\Gamma \times \{\eta_0\})$ is defined for $u \in H^{\frac{1}{2}}(\Gamma \times \{\eta_0\})$ by:

$$\tilde{v}_\delta \mapsto \langle \partial_\nu u_f - \mathrm{DtN}\, u_f, \tilde{v}_\delta \circ \mathcal{L} \rangle_{\Gamma \times \{\eta_0\}},$$

where $u_f : \Omega \setminus C^{\eta_0}_{\mathrm{ext}} \mapsto \mathbb{C}$ defined by the unique solution of: Find $u_f \in H^1_{\mathrm{loc}}(\Omega \setminus C^{\eta_0}_{\mathrm{ext}})$ such that :

$$\begin{cases} \Delta u_f + k^2 u_f = f, & \text{in } \Omega \setminus C^{\eta_0}_{\mathrm{ext}}, \\ u_f = 0, & \text{on } \Sigma_{\eta_0} \end{cases}$$

and u_f satisfies the Sommerfeld radiation.

4.1.2 Construction of the near field u_n

The near field is defined for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty$:

$$\hat{u}_n(x_\Gamma; \hat{x}, \hat{\nu}) := u_n(x_\Gamma, 0) - \sum_{i=1}^{n} \left(\mathcal{T}_0^{-1} \mathcal{T}_i \hat{u}_{n-i}\right)(x_\Gamma; \hat{x}, \hat{\nu}). \qquad (4.1.10)$$

4.1.3 Construction of an approximation of u^δ

We construct an approximation of the function $u_\delta := u^\delta \circ \mathcal{L}$ defined by:

$$u_{n,\delta} = \sum_{k=0}^{n} \delta^k u_k,$$

We assume that $m_\Gamma \geq 8$ and:

$$(\hat{\rho}, \hat{\mu}) \in C^{m_\Gamma}\Big(\Gamma; L^\infty(\hat{Y}_\infty)\Big)^2. \tag{4.1.11}$$

To assure that the exact problem is well posed we assume that:

$$\inf_{(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty} \rho(x_\Gamma; \hat{x}, \hat{\nu}) > 0. \tag{4.1.12}$$

Finally, we recall that in Chapter 3, we succeed to prove the following fundamental result:
Theorem 4.1.1. *If* $\boxed{n \leq m_\Gamma - 6}$ *and* $\operatorname{supp}(f) \subset \Omega$ *then for all* $c > 0$ *there exists* $C > 0$ *such that the following estimate holds:*

$$\|u_\delta - u_{n,\delta}\|_{H^1(\Gamma \times]c, \eta_0[)} \leq C\delta^{n+1}.$$

4.2 Effective boundary conditions

The objective of this work is to find an operator $\mathcal{Z}$ which is defined on some space of function defined on Γ and takes values in some space of function defined Γ such that if we delete from our geometry the thin coat C_δ and we replace by what we call the impedance boundary condition:

$$\partial_\nu u^\delta + \mathcal{Z}\gamma_0 u^\delta = 0,$$

where γ_0 is the classical trace operator on Γ then the new scattered field are good approximation of the exact field. The case of uniform coefficient in the thin coat has been already studied in [14].

4.3 Explicit construction of the terms $\big(\hat{u}_0(x_\Gamma; \hat{x}, \hat{\nu}), u_0(x_\Gamma, \nu)\big)$ and $\big(\hat{u}_1(x_\Gamma; \hat{x}, \hat{\nu}), u_1(x_\Gamma, \nu)\big)$.

4.3.1 The terms $\big(\hat{u}_0(x_\Gamma; \hat{x}, \hat{\nu}), u_0(x_\Gamma, \nu)\big)$

Taking $n = 0$ in (4.1.7) and (4.1.10) directly yields the following result:
Lemma 4.3.1. *The term* u_0 *is the unique solution of: Find* $u_0 \in H^1\big(\Omega_0\big)$ *such that for all* $v \in H^1\big(\Omega_0\big)$*:*

$$a_0(u_0, v) = \langle f_{\Sigma_{\eta_0}}, v \rangle_{\Gamma \times \{\eta_0\}},$$

and the term $\hat{u}_0$ *is given for* $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty$ *by* $\hat{u}_0(x_\Gamma, \hat{x}, \hat{\nu}) = u_0(x_\Gamma, 0)$.

4.3.2 The terms $\big(\hat{u}_1(x_\Gamma; \hat{x}, \hat{\nu}), u_1(x_\Gamma, \nu)\big)$

Thanks to (4.1.7) the only quantity required to compute u_1 is l_1. However, We compute here the quantity $\hat{u}_1$ because this last one is required to compute the quantity u_2. Taking $n = 1$ in (4.1.9) yields for all $x_\Gamma \in \Gamma$:

$$l_1(x_\Gamma) = \mu\Big(\big(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\hat{u}_0\big)(x_\Gamma; \cdot)\Big) - \mu\Big(\mathcal{T}_2\hat{u}_0(x_\Gamma; \cdot)\Big), \tag{4.3.13}$$

and taking $n = 1$ in (4.1.10) yields for all $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty$

$$\hat{u}_1(x_\Gamma, \hat{x}, \hat{\nu}) = u_1(x_\Gamma, 0) - \big(\mathcal{T}_0^{-1}\mathcal{T}_1 \hat{u}_0\big)(x_\Gamma, \hat{x}, \hat{\nu}). \tag{4.3.14}$$

The goal of this part is to find an explicit expression of these last terms. We emphasize that we identify for all $s \in \mathbb{R}$ the space $H^s(\Gamma)$ with the space of function in $H^s(\Gamma; \hat{Y}_\infty)$ independent of the variables $\hat{x}$ and $\hat{\nu}$ with the following injection:

$$\Big(u : x_\Gamma \in \Gamma \mapsto u(x_\Gamma)\Big) \mapsto \Big(u : (x_\Gamma; \hat{x}, \hat{\nu}) \mapsto u(x_\Gamma)\Big)$$

We see that $\mathcal{T}_0^{-1}\mathcal{T}_1 u_0(\cdot, 0)$ appears in (4.3.13) and (4.3.14). Let us detail the expression of the restriction on $H^1(\Gamma)$ of the operator $\mathcal{T}_0^{-1}\mathcal{T}_1$, more precisely of $\mathcal{T}_0^{-1}\mathcal{T}_1 u$ for $u \in H^1(\Gamma)$ that only depend of x_Γ. That will be the object of Proposition 4.3.2.

We introduce for convenience the vector fields (e_1, e_2) on Γ_{M} defined for $x_\Gamma \in \Gamma$ and $i = 1, 2$ by:

$$e_i(x_\Gamma) := (\mathrm{D}\,\psi_\Gamma(x_\Gamma))^{-1}\, \hat{e}_i \text{ if } x_\Gamma \in \Gamma_{\mathrm{M}} \quad \text{and} \quad e_i(x_\Gamma) = 0 \text{ else}, \tag{4.3.15}$$

where $\big(\hat{e}_i\big)_{i=1,2}$ is the canonical basis of $\mathbb{R}^2$. For all x_Γ in Γ_{M}, $(e_i)_{i=1,2}$ is a basis of the tangent space $T_{x_\Gamma}\Gamma$. Moreover we can introduce for $x_\Gamma \in \Gamma_{\mathrm{M}}$ the dual basis $(e^i(x_\Gamma))_{i=1,2}$ of $(e_i(x_\Gamma))_{i=1,2}$ defined as the unique element of $(T_{x_\Gamma}\Gamma)^2$ such that for all $(i, j) \in \{1, 2\}^2$:

$$(e^i(x_\Gamma), e_j(x_\Gamma)) = \delta_{ij}, \tag{4.3.16}$$

where δ_{ij} is the Kronecker symbol. For $x_\Gamma \notin \Gamma_{\mathrm{M}}$ we define these last vectors by zero. **For all that follow in this work we use in this work Einstein notation.** According to this convention, when an index variable appears twice in a single term it implies summation of that term over all the values of the index. For an example the following expression:

$$y = \sum_{i=1}^{3} c_i x_i,$$

is reduced by this convention to $y = c_i x_i$.

We introduce for $i \in \{1, 2\}$ and $x_\Gamma \in \Gamma_{\mathrm{M}}$ the function $w_i(x_\Gamma; \cdot) \in \mathbb{H}\big(\hat{Y}_\infty\big)$ as the unique solution of that we call the "cell problem": Find $w_i(x_\Gamma; \cdot) \in \mathbb{H}\big(\hat{Y}_\infty\big)$ such that for all $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ we have:

$$\int_{\hat{Y}_\infty} \Big(\hat{\rho}(\widehat{\nabla}\, w_i, \widehat{\nabla}\, v)\Big)\, (x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \int_{\hat{Y}_-} \hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{\hat{x}_i} \overline{v} d\hat{x} d\hat{\nu}, \tag{4.3.17}$$

with:

$$\int_\Sigma w_i(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} = 0.$$

Thanks to (4.1.12) this last quantity is well defined. Thanks to (4.1.11) w_i belongs to the space $C^{m_\Gamma}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big)$ because is trivial from (4.1.11) that the field of anti-linear form $\partial_{\hat{x}_i}\hat{\rho}$ defined for x_Γ and $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\langle \partial_{\hat{x}_i}\hat{\rho}(x_\Gamma; \cdot), v\rangle_{\hat{Y}_\infty} := -\int_{\hat{Y}_-} \hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{\hat{x}_i} \overline{v(\hat{x}, \hat{\nu})} d\hat{x} d\hat{\nu}$$

belongs to the space $C^{m_\Gamma}\Big(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big)$ and (4.3.17) can be rewritten as follow:

$$w_i := \big(-\delta_\Sigma\otimes\delta_\Sigma+\mathcal{T}_0\big)^{-1}\partial_{\hat{x}_i}\hat{\rho} \tag{4.3.18}$$

where $\delta_\Sigma\otimes\delta_\Sigma:\mathbb{H}\big(\hat{Y}_\infty\big)\mapsto\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ is defined for $(u,v)\in\mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\langle\delta_\Sigma\otimes\delta_\Sigma u,v\rangle_{\hat{Y}_\infty} := \int_\Sigma u d\hat{x}\overline{\int_\Sigma v d\hat{x}}.$$

Thanks to these last definitions we can state the following result:
Proposition 4.3.2. *For u in $H^1(\Gamma)$, we have for all $(x_\Gamma,\hat{x},\hat{\nu})\in\Gamma\times\hat{Y}_\infty$:*

$$\mathcal{T}_0^{-1}(\mathcal{T}_1 u)(x_\Gamma;\hat{x},\hat{\nu}) = w_i(x_\Gamma;\hat{x},\hat{\nu})\big(e^i(x_\Gamma),\nabla_\Gamma u(x_\Gamma)\big).$$

Proof. Since u does not depend of $\hat{x}$ and $\hat{\nu}$, we have from the definition of the operator $\mathcal{T}_1$ given in (4.1.4):

$$\mathcal{T}_1 u = \widehat{\operatorname{div}}\big(\hat{\rho}\,\nabla_\Gamma u\big) \text{ on } \Gamma_{\mathrm{M}}\times\hat{Y}_\infty \quad\text{and}\quad \mathcal{T}_1 u = 0 \text{ on } \Gamma\setminus\Gamma_{\mathrm{M}}\times\hat{Y}_\infty. \tag{4.3.19}$$

From (4.3.16) we have $\nabla_\Gamma u = e_i(e^i,\nabla_\Gamma u)$. Combining this last equality with (4.3.19) and using that $(e^i,\nabla_\Gamma U)$ only depend on the variable x_Γ yields:

$$\mathcal{T}_0^{-1}\mathcal{T}_1 u = \mathcal{T}_0^{-1}\big(\widehat{\operatorname{div}}(\hat{\rho}e_i)(e^i,\nabla_\Gamma u)\big) = (e^i,\nabla_\Gamma u)\mathcal{T}_0^{-1}\big(\widehat{\operatorname{div}}(\hat{\rho}e_i)\big). \tag{4.3.20}$$

Moreover, recalling the definitions of the operator $\widehat{\operatorname{div}}$ and the vector e_i given by (4.1.3) and (4.3.15) , yields for all $i\in\{1,2\}$:

$$\widehat{\operatorname{div}}(\hat{\rho}e_i) = \operatorname{div}_{\hat{x}}\big(\hat{\rho}\,\mathrm{D}\,\psi_\Gamma\,e_i\big) = \operatorname{div}_{\hat{x}}\big(\hat{\rho}\cancel{\mathrm{D}\,\psi_\Gamma(\mathrm{D}\,\psi_\Gamma)^{-1}}\hat{e}_i\big) = \partial_{\hat{x}_i}\hat{\rho}, \tag{4.3.21}$$

Combining this last equality with (4.3.20) yields $\mathcal{T}_0^{-1}\mathcal{T}_1 u = (e^i,\nabla_\Gamma u)\mathcal{T}_0^{-1}\partial_{\hat{x}_i}\hat{\rho} = (e^i,\nabla_\Gamma u)w_i$ by (4.3.18), which ends the proof. □

To simplify 4.3.13, we now seek, for u in $H^1(\Gamma)$ an explicit expression for $x_\Gamma\in\Gamma$ of the quantity:

$$\mu\Big(\big(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1 u\big)(x_\Gamma;.)\Big),$$

through the function $x_\Gamma\in\Gamma\mapsto \boldsymbol{M}_0^\rho(x_\Gamma)\in\mathcal{L}(T_{x_\Gamma}\Gamma)$ defined as follows:

- For $x_\Gamma\in\Gamma_{\mathrm{M}}$, $\boldsymbol{M}_0^\rho(x_\Gamma)$ is the unique element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ such that for all $(i,j)\in\{1,2\}^2$ we have:

$$\big(\boldsymbol{M}_0^\rho(x_\Gamma)\,e_i(x_\Gamma),e_j(x_\Gamma)\big) := \int_{\hat{Y}_\infty}\Big(\rho\hat{\nabla}w_i,\hat{\nabla}w_j\Big)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}. \tag{4.3.22}$$

- For $x_\Gamma\in\Gamma\setminus\Gamma_{\mathrm{M}}$, $\boldsymbol{M}_0^\rho(x_\Gamma):=0$.

The reason for the introduction of this tensor lies in the following form:

Proposition 4.3.3. *Let u in $H^1(\Gamma)$, one has:*

$$\forall x_\Gamma \in \Gamma,\ \mu\Big(\big(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1 u\big)(x_\Gamma;\cdot)\Big) = \operatorname{div}_\Gamma\big(\boldsymbol{M}_0^\rho\,\nabla_\Gamma\, u\big)(x_\Gamma).$$

Before proving this last result, we need to prove the following intermediate identity:

Proposition 4.3.4. *For all $i = 1,2$ and $x_\Gamma \in \Gamma_{\mathrm{M}}$, $\boldsymbol{M}_0^\rho(x_\Gamma)$ is the only element of $T_{x_\Gamma}\Gamma$ such that:*

$$\forall i \in \{1,2\},\ \boldsymbol{M}_0^\rho(x_\Gamma)\cdot e_i(x_\Gamma) = e_j(x_\Gamma)\int_{\hat{Y}_-}\Big(e^j, \hat\rho\,\widehat{\nabla}\,w_i\Big)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu.$$

Proof. Let $x_\Gamma \in \Gamma_{\mathrm{M}}$. First, we prove that, for all $(i,j) \in \{1,2\}^2$, the following identity holds:

$$\int_{\hat{Y}_\infty}\Big(\hat\rho\,\widehat{\nabla}\,w_i, \widehat{\nabla}\,w_j\Big)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu = \int_{\hat{Y}_-}\left(\hat\rho\partial_{\hat x_i} w_j\right)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu. \tag{4.3.23}$$

Indeed, we have from the definition of the operator $\widehat{\operatorname{div}}$ given by (4.1.3):

$$\int_{\hat{Y}_\infty}\Big(\hat\rho\,\widehat{\nabla}\,w_i, \widehat{\nabla}\,w_j\Big)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu = -\Big\langle\big(\widehat{\operatorname{div}}(\hat\rho\,\widehat{\nabla}\,w_i)\big)(x_\Gamma;\cdot), w_j(x_\Gamma;\cdot)\Big\rangle_{\hat{Y}_\infty},$$

and thanks to the definition of the operator $\mathcal{T}_0$ given by (4.1.4) this last equality becomes:

$$\int_{\hat{Y}_\infty}\Big(\hat\rho\,\widehat{\nabla}\,w_i, \widehat{\nabla}\,w_j\Big)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu = -\Big\langle\big(\mathcal{T}_0 w_i\big)(x_\Gamma;\cdot), w_j(x_\Gamma;\cdot)\Big\rangle_{\hat{Y}_\infty}.$$

Combining this last equality with (4.3.18) yields:

$$\begin{aligned}\int_{\hat{Y}_\infty}\Big(\hat\rho\,\widehat{\nabla}\,w_i, \widehat{\nabla}\,w_j\Big)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu &= -\left\langle\cancel{\mathcal{T}_0(x_\Gamma)\mathcal{T}_0^{-1}(x_\Gamma)}\partial_{\hat x_i}\hat\rho(x_\Gamma;\cdot), w_j(x_\Gamma;\cdot)\right\rangle_{\hat{Y}_\infty},\\ &= \int_{\hat{Y}_\infty}\hat\rho(x_\Gamma;\hat x,\hat\nu)\partial_{\hat x_i} w_j(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu.\end{aligned}$$

Moreover, thanks to the periodicity on $\hat x$ of the function $w_j(x_\Gamma;\cdot)$, we have:

$$\int_{\hat{Y}_+}\partial_{\hat x_i} w_j(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu = 0,$$

and using that $\hat\rho(x_\Gamma;\cdot) \equiv 1$ on $\hat{Y}_+$ yields (4.3.23).

Next, for all $j \in \{1,2\}$, using the definition of the operator $\widehat{\nabla}$ given by (4.1.2) and $\big(e_j(x_\Gamma), n(x_\Gamma)\big) = 0$ yields:

$$\int_{\hat{Y}_-}\Big(\big(\hat\rho\,\widehat{\nabla}\,w_i\big)(x_\Gamma;\hat x,\hat\nu), e_j(x_\Gamma)\Big)d\hat x d\hat\nu = \int_{\hat{Y}_-}\hat\rho(x_\Gamma;\hat x,\hat\nu)\left(\mathrm{D}\,\psi_\Gamma(x_\Gamma)^\dagger\nabla_{\hat x} w_i(x_\Gamma;\hat x,\hat\nu), e_j(x_\Gamma)\right)d\hat x d\hat\nu.$$

Using the definition of the vector $e_j(x_\Gamma)$ given by (4.3.15) yields that this last equality becomes:

$$\begin{aligned}\int_{\hat{Y}_-}\Big(\big(\hat\rho\,\widehat{\nabla}\,w_i\big)(x_\Gamma;\hat x,\hat\nu), e_j(x_\Gamma)\Big)d\hat x d\hat\nu &= \int_{\hat{Y}_-}\hat\rho(x_\Gamma;\hat x,\hat\nu)\left(\mathrm{D}\,\psi_\Gamma{}^\dagger(x_\Gamma)\nabla_{\hat x} w_i(x_\Gamma;\hat x,\hat\nu), \mathrm{D}\,\psi_\Gamma{}^{-1}(x_\Gamma)\hat e_j(x_\Gamma)\right)d\hat x d\hat\nu,\\ &= \int_{\hat{Y}_-}\hat\rho(x_\Gamma;\hat x,\hat\nu)\Big(\nabla_{\hat x} w_i(x_\Gamma;\hat x,\hat\nu), \cancel{\mathrm{D}\,\psi_\Gamma(x_\Gamma)\,\mathrm{D}\,\psi_\Gamma{}^{-1}}(x_\Gamma)\hat e_j(x_\Gamma)\Big)d\hat x d\hat\nu,\\ &= \int_{\hat{Y}_-}\left(\hat\rho\partial_{\hat x_j} w_i\right)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu.\end{aligned}$$

Combining this with (4.3.23) yields for all $j \in \{1,2\}$:

$$\int_{\hat{Y}_-} \Big(\big(\hat{\rho}\,\widehat{\nabla}\, w_i\big)(x_\Gamma;\hat{x},\hat{\nu}), e_j(x_\Gamma) \Big) d\hat{x} d\hat{\nu} = \left(\boldsymbol{M}_0^{\rho}(x_\Gamma)\cdot e_i(x_\Gamma), e_j(x_\Gamma)\right),$$

which ends the proof. $\square$

Before we need to recall a notion that we define in Chapter 2, :

Definition 4.3.5. *Let $m \leq m_\Gamma$ and $u : \Gamma \mapsto \mathbb{H}(\hat{Y}_\infty)$ or $u : \Gamma \mapsto \mathbb{H}(\hat{Y}_\infty)^\dagger$. We say that u satisfies the $\mathcal{P}_m^\infty$ property if there exists $d \in \mathbb{N}$, a sequence $(u_l)_{l\in\mathbb{Z}^2\setminus\{0\}} \in H^m_{0,\Gamma_\mathrm{M}}(\Gamma;\mathbb{C}_d[\hat{\nu}])$ such that:*

$$\forall (x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma\times\hat{Y}_+,\ u(x_\Gamma;\hat{x},\hat{\nu}) = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} u_l(x_\Gamma;\hat{\nu})\phi_l(x_\Gamma;\hat{x},\hat{\nu}), \tag{4.3.24}$$

where we defined the sequence of functions $(\phi_l)_{l\in\mathbb{Z}^2\setminus\{0\}}$ for $(x_\Gamma,\hat{x},\hat{\nu}) \in \Gamma\times\hat{Y}_+$ by:

$$\phi_l(x_\Gamma,\hat{x},\hat{\nu}) := e^{i2\pi l\hat{x}} e^{-2\pi\lambda_l(x_\Gamma)\hat{\nu}} \quad \textit{with} \quad \lambda_l(x_\Gamma) := |\operatorname{D}\psi_\Gamma(x_\Gamma) l|.$$

Moreover, the sequence of polynomial are required to satisfies:

$$\sum_{l\in\mathbb{Z}^2\setminus\{0\}} |l|^q \|u_l\|_{H^m(\Gamma;\mathbb{C}_d[\hat{\nu}])} < \infty.$$

In this last definition if u is not a function then (4.3.24) mean: For all $\psi \in \mathcal{D}([0,1]^2\times]0,\infty[)\cap \mathbb{H}_{\mathrm{comp}}(\hat{Y}_\infty)$:

$$\langle u(x_\Gamma;\cdot),\psi\rangle_{\mathbb{H}_{\mathrm{comp}}(\hat{Y}_\infty)^\dagger-\mathbb{H}_{\mathrm{comp}}(\hat{Y}_\infty)} = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} \int_{\hat{Y}_\infty} u_l(x_\Gamma;\hat{\nu})\phi_l(x_\Gamma;\hat{x},\hat{\nu})\overline{\psi(\hat{x},\hat{\nu})} d\hat{x} d\hat{\nu}.$$

From Proposition 2.5.15 and Proposition 2.5.16 (See Chapter 2), we recall the following result:

Proposition 4.3.6. *For all $2 \leq n \leq m_\Gamma$:*

- *For all $v \in C^m\left(\Gamma;\mathbb{H}(\hat{Y}_\infty)\right)$ satisfying the $\mathcal{P}_m^\infty$ property we have:*

  $$\forall i \geq 1,\ \mathcal{T}_i v \in C^{m-2}\left(\Gamma;\mathbb{H}(\hat{Y}_\infty)^\dagger\right),$$

  and $\mathcal{T}_i v$ satisfy the $\mathcal{P}_{m-2}^\infty$ property.

- *For all $f \in C^m\left(\Gamma;\mathbb{H}(\hat{Y}_\infty)^\dagger\right)$ if for all x_Γ, $\langle f(x_\Gamma,\cdot),1\rangle_{\hat{Y}_\infty} = 0$ then*

  $$\forall i \geq 1,\ \mathcal{T}_0^1 f \in C^m\left(\Gamma;\mathbb{H}(\hat{Y}_\infty)\right),$$

  and $\mathcal{T}_0^{-1} v$ satisfy the $\mathcal{P}_m^\infty$ property.

Proof of Proposition 4.3.3. Thanks to Proposition 4.3.2 we have:

$$\mathcal{T}_0^{-1}\mathcal{T}_1 u = w_i(e^i, \nabla_\Gamma u).$$

By using (4.1.11), we can prove that for all $i \in \{1,2\}$ $w_i \in C^{m_\Gamma}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big)$. Moreover since $\partial_{\hat{x}_i}\hat{\rho} = 0$ on $\Gamma\times]0,1[^2\times\mathbb{R}_+^*$ and $\langle\partial_{\hat{x}_i}\hat{\rho}, 1\rangle_{\hat{Y}_\infty}$ then thanks to Proposition 4.3.6, we have that w_i satisfies $\mathcal{P}_{m_\Gamma}^\infty$ property. Thus thanks to Proposition 4.3.6 we deduce that: for all $x_\Gamma \in \Gamma$ the quantity $\big(\mathcal{T}_1(w_i(e^i, \nabla_\Gamma u))\big)(x_\Gamma;\cdot)$ belong to the space $\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$. Therefore thank to (4.1.6) we have:

$$\mu\Big(\big(\mathcal{T}_1(w_i(e^i, \nabla_\Gamma u))\big)(x_\Gamma;\cdot)\Big) = \big\langle\big(\mathcal{T}_1(w_i(e^i, \nabla_\Gamma u))\big)(x_\Gamma;\cdot), 1\big\rangle_{\hat{Y}_\infty}$$

Moreover using the definition of $\mathcal{T}_1$ given by (4.1.4) and the definition of the operator $\widehat{\operatorname{div}}$ given by (4.1.3) yields:

$$\mu\Big(\big(\mathcal{T}_1(w_i(e^i, \nabla_\Gamma u))\big)(x_\Gamma;\cdot)\Big) = \operatorname{div}_\Gamma\left(\big(e^i(x_\Gamma), \nabla_\Gamma u(x_\Gamma)\big)\int_{\hat{Y}_\infty}\Big(\hat{\rho}\,\widehat{\nabla}\, w_i\Big)(x_\Gamma;\cdot)d\hat{x}d\hat{\nu}\right),$$

and using Proposition 4.3.4 and (4.3.16) this becomes:

$$\mu\Big(\big(\mathcal{T}_1(w_i(e^i, \nabla_\Gamma u))\big)(x_\Gamma;\cdot)\Big) = \operatorname{div}_\Gamma\Big(\big(\boldsymbol{M}_0^\rho\cdot e_i(e^i, \nabla_\Gamma u)\big)(x_\Gamma)\Big) = \operatorname{div}_\Gamma\left(\boldsymbol{M}_0^\rho(x_\Gamma)\cdot\nabla_\Gamma u(x_\Gamma)\right).$$

This conclude the proof. □

Let us introduce the averages $\overline{\rho}_0, \overline{\mu}_0 : \Gamma \mapsto \mathbb{R}$ on cells of quantities $\hat{\rho}$ and $\hat{\mu}$ defined for x_Γ by:

$$\overline{\rho}_0(x_\Gamma) := \int_{\hat{Y}_-}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} \quad\text{and}\quad \overline{\mu}_0(x_\Gamma) := \int_{\hat{Y}_-}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}. \qquad (4.3.25)$$

Then, we have the following result:

Proposition 4.3.7. *For all $u \in H^1(\Gamma)$ independent of $(\hat{x},\hat{\nu})$ we have for all x_Γ:*

$$\mu\Big(\big(\mathcal{T}_2 u\big)(x_\Gamma;\cdot)\Big) = \operatorname{div}_\Gamma\left(\overline{\rho}_0(x_\Gamma)\,\nabla_\Gamma u(x_\Gamma)\right) + k^2\overline{\mu}_1 u(x_\Gamma). \qquad (4.3.26)$$

Proof. Since u does not depend on the variable $(\hat{x},\hat{\nu})$ we have:

$$\mathcal{T}_{k,0}u = \operatorname{div}_\Gamma\Big(\rho\,\nabla_\Gamma u\Big) + k^2\mu u, \quad \mathcal{T}_{k,1}u = \widehat{\operatorname{div}}\left(\hat{\nu}\rho\mathcal{C}^{(1)}\,\nabla_\Gamma u\right) \text{ and } \mathcal{T}_{k,2}u = 0,$$

which leads combined with (4.1.4) to the following decomposition:

$$\mathcal{T}_2 u = A + B, \qquad (4.3.27)$$

where A, B are given by:

$$\begin{cases} A := \Delta_\Gamma u + k^2 u, \\ B := \operatorname{div}_\Gamma\Big((\hat{\rho}-1)\,\nabla_\Gamma u\Big) + k^2(\hat{\mu}-1)u + \widehat{\operatorname{div}}\Big(\hat{\nu}\rho\mathcal{C}^{(1)}\,\nabla_\Gamma u\Big), \end{cases}$$

and $\big(A(x_\Gamma), B(x_\Gamma;\cdot)\big) \in \mathbb{C}[\hat{\nu}] \times \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ for all x_Γ. On the on hand we have:

$$\int_{-1}^{0} A\big(x_\Gamma\big)\mathrm{d}\hat{\nu} = \Delta_\Gamma u(x_\Gamma) + k^2 u(x_\Gamma). \tag{4.3.28}$$

On the other hand we have:

$$\begin{aligned}
\langle B(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty} =& \Big\langle \operatorname{div}_\Gamma \big((\hat{\rho}(x_\Gamma;\cdot)-1)\,\nabla_\Gamma\, u(x_\Gamma)\big),1\Big\rangle_{\hat{Y}_\infty} \\
&+\Big\langle k^2(\hat{\mu}(x_\Gamma;\cdot)-1)u(x_\Gamma),1\Big\rangle_{\hat{Y}_\infty} \\
&\cancel{+\langle \widehat{\operatorname{div}}\,\hat{\nu}\rho(x_\Gamma;\cdot)\mathcal{C}^{(1)}\,\nabla_\Gamma\, u(x_\Gamma),1\rangle_{\hat{Y}_\infty}}, \\
&= \operatorname{div}_\Gamma\Big(\big(\int_{\hat{Y}_-}(\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})-1)d\hat{x}d\hat{\nu}\big)\,\nabla_\Gamma\, u(x_\Gamma)\Big) \\
&+k^2\left(\int_{\hat{Y}_-}(\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})-1)d\hat{x}d\hat{\nu}\right)u(x_\Gamma),
\end{aligned}$$

which leads combined with the definition of $\overline{\rho}_0$ and $\overline{\mu}_0$ given by (4.3.25) to

$$\langle B(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty} = \operatorname{div}_\Gamma\Big((\overline{\rho}_0(x_\Gamma)-1)\,\nabla_\Gamma\, u(x_\Gamma)\Big) + k^2(\overline{\mu}_0(x_\Gamma)-1)u(x_\Gamma).$$

Thanks to this last equality, (4.3.28) and the decomposition (4.3.27), we can apply the definition of μ given by (4.1.6) which yields (4.3.26) and so ends the proof. □

Finally, we introduce the tensor field:

$$\boldsymbol{\rho}^0_{eff} = \overline{\rho}_0 I - \boldsymbol{M}^\rho_0, \tag{4.3.29}$$

the operator $\mathcal{Z}_1 : H^1(\Gamma) \mapsto H^{-1}(\Gamma)$ defined for $u \in H^1(\Gamma)$:

$$\mathcal{Z}_1 u := \operatorname{div}_\Gamma\big(\boldsymbol{\rho}^0_{eff}\,\nabla_\Gamma\, u\big) + k^2\hat{\mu}_0 u,$$

and the trace operator $\gamma_\Gamma : H^1\big(\Omega_0\big) \mapsto H^{\frac{1}{2}}\big(\Gamma\big)$ in order to state the following result:

Lemma 4.3.8. *The term u_1 is the unique solution of the problem: Find $u_1 \in H^1\big(\Omega_0\big)$ such that for all $v \in H^1\big(\Omega_0\big)$:*

$$a_0(u_1,v) = \langle \mathcal{Z}_1\gamma_\Gamma u_0, v\rangle_{\Gamma\times\{0\}},$$

and the term $\hat{u}_1$ is given for $(x_\Gamma,\hat{x},\hat{\nu}) \in \Gamma\times\hat{Y}_\infty$ by:

$$\hat{u}_1(x_\Gamma;\hat{x},\hat{\nu}) = u_1(x_\Gamma,0) - w_i(x_\Gamma;\hat{x},\hat{\nu})(e^i(x_\Gamma),\nabla_\Gamma\, u_0(x_\Gamma,0)).$$

4.4 Explicit construction of the term $u_2(x_\Gamma,\nu)$.

Thanks to (4.1.7), the only quantity we need to compute the term u_2 is l_2.

4.4.1 Decomposition of the term l_2

Dues to the complexity of the quantity l_2, one need to introduce for convenience the following quantity defined for $x_\Gamma \in \Gamma$ by

$$\begin{cases} l_2^\alpha(x_\Gamma) := -\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big), \\ l_2^\beta(x_\Gamma) := \mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_2\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) + \mu\Big(\left(\mathcal{T}_2\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big), \\ l_2^\gamma(x_\Gamma) := -\mu\Big(\left(\mathcal{T}_3\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big), \\ l_2^\delta(x_\Gamma) := \mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_1\right)(x_\Gamma;\cdot)\Big) - \mu\Big(\left(\mathcal{T}_2\gamma_\Gamma u_1\right)(x_\Gamma;\cdot)\Big). \end{cases} \tag{4.4.30}$$

Indeed, we have the following result:
Proposition 4.4.1. *The quantity l_2 can be rewritten as follow:*

$$l_2 = l_2^\alpha + l_2^\beta + l_2^\gamma + l_2^\delta, \tag{4.4.31}$$

Proof. Taking $n = 2$ in (4.1.9) yields that the term l_2 is given for $x_\Gamma \in \Gamma$ by

$$l_2(x_\Gamma) = \sum_{(i,j)\in\{1,2\}^2} \mu\Big(\left(\mathcal{T}_i\mathcal{T}_0^{-1}\mathcal{T}_j\hat{u}_{2+1-i-j}\right)(x_\Gamma;\cdot)\Big) - \mu\Big(\left(\mathcal{T}_2\gamma_\Gamma u_1\right)(x_\Gamma;\cdot)\Big) + l_2^\gamma(x_\Gamma).$$

The sum appearing in this last equality can be rewritten as follow:

$$\sum_{(i,j)\in\{1,2\}^2} \mu\Big(\left(\mathcal{T}_i\mathcal{T}_0^{-1}\mathcal{T}_j\hat{u}_{2+1-i-j}\right)(x_\Gamma;\cdot)\Big) = \mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\hat{u}_1\right)(x_\Gamma;\cdot)\Big) + l_2^\beta(x_\Gamma) \tag{4.4.32}$$

Moreover from (4.3.13) we have $\hat{u}_1 = \gamma_\Gamma u_1 - \mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0$ which yields:

$$\begin{aligned} \mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\hat{u}_1\right)(x_\Gamma;\cdot)\Big) &= \mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\left(\gamma_\Gamma u_1 - \mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)\right)(x_\Gamma;\cdot)\Big), \\ &= l_2^\alpha(x_\Gamma) + \mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_1\right)(x_\Gamma;\cdot)\Big). \end{aligned}$$

Finally combining this last equality with (4.4.32) ends the proof of our result. ☐

Dues to the complexity of this last expressions we have chosen to explain the computation of terms $l_\alpha, l_\beta, l_\gamma$ and l_δ into several parts.

4.4.2 The term $l_2^\delta(x_\Gamma)$

Since u_1 only depend of the variable x_Γ, we can apply Proposition 4.3.3 and Proposition 4.3.7 which leads to:

$$l_2^\delta = -\mathcal{Z}_1\gamma_\Gamma u_1.$$

4.4.3 The term $l_2^\gamma(x_\Gamma)$

Let us define the 1-average on cells of quantities $\hat{\rho}$ and $\hat{\mu}$ for $x_\Gamma \in \Gamma$ by:

$$\overline{\rho}_1(x_\Gamma) := \int_{\hat{Y}_-} 2\hat{\nu}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} \quad \text{and} \quad \overline{\mu}_1(x_\Gamma) := \int_{\hat{Y}_-} 2\hat{\nu}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},$$

in order to state the following result:

Proposition 4.4.2. *The term l_2^γ can be rewritten as follow:*

$$l_2^\gamma = \mathrm{div}_\Gamma\left(\overline{\rho}_1(H-R)\,\nabla_\Gamma\,\gamma_\Gamma u_0\right) + k^2\overline{\mu}_1 H\gamma_\Gamma u_0.$$

Proof. Since $\hat{u}_0$ only depend of x_Γ we have from (4.1.4):

$$\mathcal{T}_3\hat{u}_0 = \underbrace{\widehat{\mathrm{div}}\left(\hat{\nu}^2\hat{\rho}\mathcal{C}^{(2)}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)}_{Q_1} + \underbrace{\mathrm{div}_\Gamma\left(\hat{\nu}\hat{\rho}\mathcal{C}^{(1)}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right) + k^2\hat{\nu}\hat{\mu}c^{(1)}\gamma_\Gamma u_0}_{Q_2}. \tag{4.4.33}$$

We have:

$$Q_1 = \widehat{\mathrm{div}}\left(\hat{\nu}^2(\hat{\rho}-1)\mathcal{C}^{(2)}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right) + \cancel{\widehat{\mathrm{div}}\left(\hat{\nu}^2\mathcal{C}^{(2)}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)}, \tag{4.4.34}$$

Therefore for all $x_\Gamma \in$ we have $Q_1(x_\Gamma;\cdot) \in \mathbb{H}(\hat{Y}_\infty)^\dagger$ and then according to (4.1.6) we have for all $x_\Gamma \in \Gamma$

$$\begin{aligned}\mu\Big(Q_1(x_\Gamma;\cdot)\Big) &= \Big\langle\Big(\widehat{\mathrm{div}}\left(\hat{\nu}^2(\hat{\rho}-1)\mathcal{C}^{(2)}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)\Big)(x_\Gamma;\cdot),1\Big\rangle_{\hat{Y}_\infty},\\ &= -\int_{\hat{Y}_\infty}\left(\hat{\nu}^2(\hat{\rho}-1)\mathcal{C}^{(2)}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\cdot\cancel{\widehat{\nabla}1}d\hat{x}d\hat{\nu} = 0.\end{aligned}$$

We proceed as the same way as the end of proof of Proposition 4.3.7 to show that:

$$\mu\Big(Q_2(x_\Gamma;\cdot)\Big) = -\frac{\mathrm{div}_\Gamma\left(\overline{\rho}_1\mathcal{C}^{(1)}\,\nabla_\Gamma\,u_0(x_\Gamma,0)\right) + k^2\overline{\mu}_1 c^{(1)}(x_\Gamma)u_0(x_\Gamma,0)}{2}.$$

Combining this last equation with Proposition 2.5.1, (4.4.34) and (4.4.33) yields the desired result: □

4.4.4 The others terms $l_2^\alpha(x_\Gamma), l_2^\beta(x_\Gamma)$

One of the biggest difficulty we encountered for the computation of the term l_2 is the apparition of 1 and 3 order tangential differential operators in the expression of the terms l_2^α and l_2^β when we compute them with straightforward development. However, we remark that for all smooth function $a : \mathbb{R} \mapsto \mathbb{R}$ the operator $T_a : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ defined for $u \in H^1(\mathbb{R})$ by $T_a u := l \cdot \partial_x u$ satisfies the following identity:

$$\forall u \in H^1(\mathbb{R}), \quad \mathrm{Re}Tu = -\frac{\partial_x a}{2}u,$$

where we defined for any linear operator A the operator:

$$\mathrm{Re}A := \frac{A + A^\dagger}{2}. \tag{4.4.35}$$

4.4.4.1 The real part of a an odd differential operator and the $\star$ operator

The following result is a first extension of (4.4.35) for vector field $a : \Gamma \mapsto \mathbb{R}^3$:

Proposition 4.4.3. *Let $a \in C^{m_\Gamma}(\Gamma;\mathbb{R}^3)$ be a vector field and define $T_a : H^1(\Gamma) = D(T_a) \subset L^2(\Gamma) \mapsto L^2(\Gamma)$ for $u \in H^1(\Gamma)$ by:*

$$T_a u := (a, \nabla_\Gamma u),$$

then for all $u \in L^2(\Gamma)$ we have:

$$\mathrm{Re}(T_a)u := \frac{T_a + T_a^\dagger}{2} u = -\frac{\mathrm{div}_\Gamma\, a}{2} u.$$

Proof. Thanks to Leibniz formula we have:

$$T_a^\dagger u = -\,\mathrm{div}_\Gamma\,(au) = -(\nabla_\Gamma u, a) - \mathrm{div}_\Gamma(a)u = -T_a u - \mathrm{div}_\Gamma(a)u,$$

which directly yields our stated result. □

We want a similar result for 3 order differential tangential operators that take the forms:

$$T_{\alpha,\beta} u := \mathrm{div}_\Gamma\,\big(\beta\, \mathrm{div}_\Gamma(\alpha\, \nabla_\Gamma u)\big),$$

where α and β are respectively a tensor field and vector field with at least C^1 regularities such that these quantities vanish on $\Gamma \setminus \Gamma_{\mathrm{M}}$. From these last quantities we define a new tensor field named $\alpha \star \beta$. First this quantity is defined by 0 on $\Gamma \setminus \Gamma_{\mathrm{M}}$. Now let us explain the way we defined this quantity on Γ_{M}. Moreover we define for smooth quantity u defined on Γ the following notation:

$$\forall x_\Gamma \in \Gamma_{\mathrm{M}},\ \partial_{x_k} u(x_\Gamma) := \partial_{x_k}\big(u \circ \psi_\Gamma{}^{-1}\big)\big(\psi_\Gamma(x_\Gamma)\big) \quad \text{and} \quad \forall x_\Gamma \in \Gamma \setminus \Gamma_{\mathrm{M}}\, \partial_{x_k} u(x_\Gamma) := 0. \tag{4.4.36}$$

We define for $(i,j) \in \{1,2\}^2$ the quantities:

$$\beta^i := (\beta, e^i) \quad \text{and} \quad \alpha^{ij} := (\alpha e^j, e^i).$$

Thus the tensor field $\alpha \star \beta$ is defined on Γ_{M} by the unique tensor field such that for all $(i,j) \in \{1,2\}^2$:

$$(\alpha \star \beta e^j, e^i) := \frac{1}{\sqrt{g}}\left[\partial_{x_k}\big(\sqrt{g}(\alpha^{jk}\beta^i - \beta^k\alpha^{ji})\big) + \sqrt{g}\alpha^{ki}\partial_{x_k}\beta^j - \beta^i\partial_{x_k}(\sqrt{g}\alpha^{kj})\right], \tag{4.4.37}$$

and this last tensor is defined by zero on $\Gamma \setminus \Gamma_{\mathrm{M}}$. The reason that we introduced $\star$ lies in the following forms:

Proposition 4.4.4. *The real part of $T_{\alpha,\beta}$ operator is given for $u \in H^3(\Gamma)$ by:*

$$\mathrm{Re}(T_{\alpha,\beta})u = \frac{1}{2}\,\mathrm{div}_\Gamma\,\big(\alpha \star \beta\, \nabla_\Gamma u\big).$$

Proof. The Einstein summation convention is taken for whole this proof. We recall that the followings expression of ∇_Γ and div_Γ holds on $\Gamma \setminus \Gamma_{\mathrm{M}}$:

$$\forall u,\ \nabla_\Gamma u = e^i \partial_{x_i} u \quad \text{and} \quad \forall \boldsymbol{u}\, \mathrm{div}_\Gamma(u^i e_i) = \frac{1}{\sqrt{g}}\partial_{x_i}\left(\sqrt{g}u^i\right), \tag{4.4.38}$$

where $\boldsymbol{u}$ is a regular tangential field and $(u_i)_{1\leq i\leq 2}$ is the unique scalar field such that $\boldsymbol{u} = u^i e_i$. Let $u \in H^3(\Gamma)$. Thanks to (4.4.38) we have:

$$T_{\alpha,\beta} u = \frac{1}{\sqrt{g}} \partial_{x_k} \Big(\beta^k \partial_{x_j} \big(\sqrt{g} \alpha^{ji} \partial_{x_i} u \big) \Big) \quad \text{and} \quad T^\dagger_{\alpha\beta} u = -\frac{1}{\sqrt{g}} \Big(\partial_{x_i} \sqrt{g} \alpha^{ji} \big(\partial_{x_j} \beta^k \partial_{x_k} u \big) \Big).$$

Leibniz formula yields that for all u:

$$\begin{cases} \sqrt{g} T^\dagger_{\alpha\beta} u = \sqrt{g} \beta^k \alpha^{ji} \partial_{x_k} \big(\partial_{x_j} (\partial_{x_i} u) \big) + \partial_{x_k} (\sqrt{g} \beta^k \alpha^{ji}) \partial_{x_j} (\partial_{x_i} u) + \partial_{x_k} \big(\beta^k \partial_{x_j} (\sqrt{g} \alpha^{ji}) \partial_{x_i} u \big) \\ -\sqrt{g} T^\dagger_{\alpha,\beta} u = \sqrt{g} \alpha^{ji} \beta^k \partial_{x_i} \big(\partial_{x_j} (\partial_{x_k} u) \big) + \partial_{x_i} (\sqrt{g} \alpha^{ji} \beta^k) \partial_{x_j} (\partial_{x_k} u) + \partial_{x_i} \big(\sqrt{g} \alpha^{ji} \partial_{x_j} \beta^k (\partial_{x_k} u) \big) \end{cases}$$

Doing the difference between these last equalities yields:

$$\begin{aligned} \big(\sqrt{g} T_{\alpha,\beta} + \sqrt{g} T^\dagger_{\alpha,\beta} \big) u &= \partial_{x_i} (\sqrt{g} \alpha^{ji} \beta^k) \partial_{x_j} (\partial_{x_k} u) - \partial_{x_k} (\sqrt{g} \beta^k \alpha^{ji}) \partial_{x_j} (\partial_{x_i} u) \\ &\quad + \partial_{x_i} \big(\partial_{x_j} (\sqrt{g} \alpha^{ji} \beta^k) \partial_{x_k} u \big) - \partial_{x_k} \big(\beta^k \partial_{x_j} (\sqrt{g} \alpha^{ji}) \partial_{x_i} u \big), \\ &= \partial_{x_k} \big(\partial_{x_i} (\sqrt{g} \alpha^{ji} \beta^k) \partial_{x_j} u \big) - \partial_{x_i} \big(\partial_{x_k} (\sqrt{g} \beta^k \alpha^{ji}) \partial_{x_j} u \big) \\ &\quad + \partial_{x_i} \big(\partial_{x_j} (\sqrt{g} \alpha^{ji} \beta^k) \partial_{x_k} u \big) - \partial_{x_k} \big(\beta^k \partial_{x_j} (\sqrt{g} \alpha^{ji}) \partial_{x_i} u \big), \end{aligned}$$

and an index permutation yields:

$$\big(T_{\alpha,\beta} + T^\dagger_{\alpha,\beta} \big) u = \frac{1}{\sqrt{g}} \partial_{x_i} \Big(\big[\partial_{x_k} \big(\sqrt{g} (\alpha^{jk} \beta^i - \beta^k \alpha^{ji}) \big) + \sqrt{g} \alpha^{ki} \partial_{x_k} \beta^j - \beta^i \partial_{x_k} (\sqrt{g} \alpha^{kj}) \big] \, \partial_{x_j} u \Big).$$

Thanks to the definition of $\star$ given in (4.4.37) and the ones of the operator $\operatorname{div}_\Gamma$ and ∇_Γ given in (4.4.38), this becomes:

$$\big(T_{\alpha,\beta} + T^\dagger_{\alpha,\beta} \big) u = \frac{1}{\sqrt{g}} \partial_{x_i} \Big(\sqrt{g} (\alpha \star \beta)^{ij} \partial_{x_j} u \Big) = \operatorname{div}_\Gamma \big(\alpha \star \beta e^i (e_j, \nabla_\Gamma u) \big) = \operatorname{div}_\Gamma \left(\alpha \star \beta \, \nabla_\Gamma u \right),$$

which ends the proof. □

4.4.4.2 The term $l_2^\alpha(x_\Gamma)$

Let us prove the following intermediate result:

Proposition 4.4.5. *Let $L \in H^1(\Gamma; \mathbb{H}(\hat{Y}_\infty)^\dagger)$ satisfying the $\mathcal{P}_1^\infty$ property. For all $x_\Gamma \in \Gamma$ we have:*

$$\mu\Big(\big(\mathcal{T}_1 \mathcal{T}_0^{-1} L \big) \, (x_\Gamma; \cdot) \Big) = -\operatorname{div}_\Gamma \Big(e^i(x_\Gamma) \big\langle L(x_\Gamma; \cdot), w_i(x_\Gamma; \cdot) \big\rangle_{\hat{Y}_\infty} \Big),$$

and for all $P \in H^1(\Gamma; \mathbb{C}[\hat{\nu}])$:

$$\mu\Big(\big(\mathcal{T}_1 \mathcal{T}_0^{-1} P \big) \, (x_\Gamma; \cdot) \Big) = -\operatorname{div}_\Gamma \left(e^i(x_\Gamma) \int_{\hat{Y}_-} w_i(x_\Gamma; \hat{x}, \hat{\nu}) P(x_\Gamma; \hat{\nu}) d\hat{x} d\hat{\nu} \right).$$

Proof. First prove the result for L. We assume first that for all $x_\Gamma \in \Gamma$ we have

$$\langle L(x_\Gamma; \cdot), 1 \rangle_{\hat{Y}_\infty} = 0 \tag{4.4.39}$$

Thus using Proposition 4.3.6 yields that for all x_Γ $\big((\mathcal{T}_1\mathcal{T}_0^{-1})L\big)(x_\Gamma;\cdot)\in\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ satisfies P_0^∞ property which leads to

$$\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot)\Big)=\Big\langle\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot),1\Big\rangle_{\hat{Y}_\infty}.$$

Thanks to the definition of the operator $\mathcal{T}_1$ given in (4.1.4) and the one of the operator $\widehat{\mathrm{div}}$ given in (4.1.3) yields:

$$\begin{aligned}\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot)\Big)&=\Big\langle\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot),1\Big\rangle_{\hat{Y}_\infty},\\&=\mathrm{div}_\Gamma\left(e^i(x_\Gamma)\int_{\hat{Y}_\infty}\Big(\big(\hat{\rho}\,\widehat{\nabla}\,\mathcal{T}_0^{-1}L\big)(x_\Gamma;\hat{x},\hat{\nu}),e_i(x_\Gamma)\Big)d\hat{x}d\hat{\nu}\right),\\&=-\,\mathrm{div}_\Gamma\left(e^i(x_\Gamma)\Big\langle\widehat{\mathrm{div}}\left(\hat{\rho}(x_\Gamma;\cdot)e_i(x_\Gamma)\right),\mathcal{T}_0^{-1}L(x_\Gamma;\cdot)\Big\rangle_{\hat{Y}_\infty}\right).\end{aligned}$$

Therefore using $\widehat{\mathrm{div}}(\hat{\rho}e_i)=\partial_{\hat{x}_i}\hat{\rho}$(see the proof of Proposition 4.3.2) yields that for all $x_\Gamma\in\Gamma_\mathrm{M}$:

$$\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot)\Big)=-\,\mathrm{div}_\Gamma\left(e^i(x_\Gamma)\left\langle\partial_{\hat{x}_i}\hat{\rho}(x_\Gamma;\cdot),\mathcal{T}_0^{-1}(x_\Gamma)L(x_\Gamma;\cdot)\right\rangle_{\hat{Y}_\infty}\right).\qquad(4.4.40)$$

Since the quantity $L(x_\Gamma;\cdot)$ are assumed to belongs to $\mathbb{H}\big(\hat{Y}_\infty\big)$, the quantity $\mathcal{T}_0^{-1}(x_\Gamma)L(x_\Gamma;\cdot)$ is given by $\big(-\delta_\Sigma\otimes\delta_\Sigma+\mathcal{T}_0(x_\Gamma)\big)^{-1}L(x_\Gamma;\cdot)$ where $\big(-\delta_\Sigma\otimes\delta_\Sigma+\mathcal{T}_0(x_\Gamma)\big)^{-1}$ is seen as a continuous linear operator from $\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ into $\mathbb{H}\big(\hat{Y}_\infty\big)$. Moreover this last operator is self-adjoint because $\hat{\rho}$ takes values in $\mathbb{R}$ and hence (4.4.40) becomes:

$$\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot)\Big)=-\,\mathrm{div}_\Gamma\left(e^i(x_\Gamma)\Big\langle L(x_\Gamma;\cdot),\big(-\delta_\Sigma\otimes\delta_\Sigma+\mathcal{T}_0(x_\Gamma)\big)^{-1}\partial_{\hat{x}_i}\hat{\rho}(x_\Gamma;\cdot)\Big\rangle_{\hat{Y}_\infty}\right).$$

Thus using the definition of the function $w_i(x_\Gamma;\cdot)$ given in (4.3.18) yields:

$$\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}L(x_\Gamma;\cdot)\right)(x_\Gamma;\cdot)\Big)=-\,\mathrm{div}_\Gamma\Big(e_i(x_\Gamma)\langle L(x_\Gamma;\cdot),w_i(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty}\Big).$$

Therefore we success to prove the following implication: If for all $x_\Gamma\in\Gamma$ we have (4.4.39) then :

$$\mu\Big(\big(\mathcal{T}_1\mathcal{T}_0^{-1}L\big)(x_\Gamma;\cdot)\Big)=-\,\mathrm{div}_\Gamma\Big(e_i(x_\Gamma)\langle L(x_\Gamma;\cdot),w_i(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty}\Big).\qquad(4.4.41)$$

Now, we assume that:

$$\exists l\in H^1(\Gamma)\ \mathrm{sq}\ L=l\cdot\delta_\Sigma,\qquad(4.4.42)$$

where $\delta_\Sigma\in\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$ is defined for $u\in\mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\langle\delta_\Sigma,u\rangle_{\hat{Y}_\infty}:=\overline{\int_\Sigma u\,d\hat{x}}.$$

Thanks to Proposition 2.5.5(See Chapter 2) we have $\mathcal{T}_0^{-1}L=\hat{\nu}_+\cdot l$. Thus on the one hand we have:

$$\Big(\mathrm{div}_\Gamma\,\hat{\rho}\,\widehat{\nabla}+\widehat{\mathrm{div}}\,\hat{\rho}\,\nabla_\Gamma\Big)\cdot\hat{\nu}_+\cdot l=\cancel{\mathrm{div}_\Gamma\left(l\cdot\hat{\rho}\chi_{\nu>0}n\right)}+\widehat{\mathrm{div}}\left(\hat{\nu}_+\hat{\rho}\,\nabla_\Gamma l\right)=\cancel{\partial_{\hat{\nu}}\left((n,\hat{\nu}_+\hat{\rho}\,\nabla_\Gamma l)\right)}=0.$$

On the other hand we have the following decomposition:

$$\widehat{\operatorname{div}}\left(\hat{\rho}\hat{\nu}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{\nu}_+\right) = \widehat{\operatorname{div}}\left(\hat{\nu}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{\nu}_+\right) = \partial_{\hat{\nu}}\left(\hat{\nu}_+ H\right) = 1_{\mathbb{R}_+}(\hat{\nu})\cdot H = 1 - 1_{\mathbb{R}_-}(\hat{\nu})\cdot H,$$

which leads to for all $x_\Gamma \in \Gamma$:

$$\mu\Big(\Big(\widehat{\operatorname{div}}\left(\hat{\rho}\hat{\nu}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{\nu}_+\right)\Big)(x_\Gamma;\cdot)\Big) = \cancel{\int_{-1}^{0} H(x_\Gamma)d\hat{\nu}} - \cancel{\langle 1_{\mathbb{R}_-}(\hat{\nu})\cdot H(x_\Gamma), 1\rangle_{\hat{Y}_\infty}} = 0.$$

Therefore we get $\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}L\right)(x_\Gamma;\cdot)\Big) = 0$. Moreover, we recall that (4.3.17) states that for all $i \in \{1,2\}$ we have $\langle \delta_\Sigma, w_i(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty} = 0$ which leads to:

$$\mu\Big(\Big(\widehat{\operatorname{div}}\left(\hat{\rho}\hat{\nu}\mathcal{C}^{(1)}\,\widehat{\nabla}\,\hat{\nu}_+\right)\Big)(x_\Gamma;\cdot)\Big) = \operatorname{div}_\Gamma\left(e_i(x_\Gamma)\langle w_i(x_\Gamma;\cdot), L(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty}\right) = 0.$$

Therefore we success to prove the following implication: (4.4.42) $\Rightarrow$ (4.4.41).

Now we can combine this last implication with the first implication we proved: (4.4.39) $\Rightarrow$ (4.4.41). Indeed, we have for all $L \in H^1(\Gamma;\mathbb{H}(\hat{Y}_\infty))$ the following decomposition holds of L:

$$\forall x_\Gamma \in \Gamma,\ L(x_\Gamma) = \underbrace{\langle L(x_\Gamma), 1\rangle_{\hat{Y}_\infty}\cdot\delta_\Sigma}_{A_1} + \underbrace{\left(L(x_\Gamma) - \langle L(x_\Gamma), 1\rangle_{\hat{Y}_\infty}\cdot\delta_\Sigma\right)}_{A_2},$$

where A_1 and A_2 respectively satisfy (4.4.39) and (4.4.42). Then these two last quantities both satisfies (4.4.41) and we can conclude thanks to the the linearity of $\mu\Big(.\Big)$ and so ends the proof of the result for the quantity L.

Now let us investigate the case for the quantity P. Introduce the quantity:

$$Q := P - \mathcal{T}_0\cdot d_{\hat{\nu}}^{-2}P, \tag{4.4.43}$$

where we recall that for all x_Γ, $d_{\hat{\nu}}^{-2}P(x_\Gamma;\cdot)$ is defined as the unique solution of:

$$\begin{cases} \dfrac{d^2}{d\nu^2}\left(d_{\hat{\nu}}^{-2}P(x_\Gamma;\nu)\right) = P, \\ \qquad d_{\hat{\nu}}^{-2}P(x_\Gamma;0) = 0, \\ \dfrac{d}{d\nu}\left(d_{\hat{\nu}}^{-2}P(x_\Gamma;-1)\right) = 0. \end{cases}$$

Thanks to Corollary 2.5.7 (See Chapter 2) we have that Q belongs to $H^1(\Gamma;\mathbb{H}(\hat{Y}_\infty)^\dagger)$ which leads from what we have shown for the case of L to:

$$\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}Q\right)(x_\Gamma;\cdot)\Big) = -\operatorname{div}_\Gamma\left(e^i(x_\Gamma)\langle Q(x_\Gamma;\cdot), w_i(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty}\right) \tag{4.4.44}$$

Now let us prove that:

$$\begin{cases} \mu\Big(\left(\mathcal{T}_1 d_{\hat{\nu}}^{-2}P\right)(x_\Gamma;\cdot)\Big) = \operatorname{div}_\Gamma\left(e^i(x_\Gamma)\displaystyle\int_{\hat{Y}_-} P(x_\Gamma;\hat{\nu})w_i(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right) \\ \qquad - \operatorname{div}_\Gamma\left(e^i(x_\Gamma)\,\langle Q(x_\Gamma;\hat{x},\hat{\nu}), w_i(x_\Gamma;\hat{x},\hat{\nu})\rangle_{\hat{Y}_\infty}\right) \\ \qquad = 0. \end{cases} \tag{4.4.45}$$

To prove this last equality, we first prove the following equality:

$$\mu\Big(\widehat{\operatorname{div}}\left(\hat\rho\hat\nu\mathcal{C}^{(1)}\,\widehat{\nabla}\, d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot)\Big) + \mu\Big(\widehat{\operatorname{div}}\left(\hat\rho\,\nabla_\Gamma\, d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot)\Big) = 0. \tag{4.4.46}$$

Indeed we have $\widehat{\operatorname{div}}\left((\hat\rho-1)\hat\nu\mathcal{C}^{(1)}\,\widehat{\nabla}\, d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot)+\widehat{\operatorname{div}}\left((\hat\rho-1)\,\nabla_\Gamma\, d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot) \in \mathbb{H}\big(\hat Y_\infty\big)$, which leads combined with the definition of the operator $\widehat{\operatorname{div}}$ given by (4.1.3) to

$$\mu\Big(\widehat{\operatorname{div}}\left((\hat\rho-1)\hat\nu\mathcal{C}^{(1)}\,\widehat{\nabla}\, d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot) + \widehat{\operatorname{div}}\left((\hat\rho-1)\,\nabla_\Gamma\, d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot)\Big) = 0 \tag{4.4.47}$$

Moreover, on the one hand we have for all $x_\Gamma\in\Gamma$ $\left(\nabla_\Gamma\, d_{\hat\nu}^{-2}P\right)(x_\Gamma)\in T_{x_\Gamma}\Gamma$ and using that this last quantity only depend on $\hat\nu$ yields

$$\widehat{\operatorname{div}}\left(\hat\rho\,\nabla_\Gamma\, d_{\hat\nu}^{-2}P\right) = 0. \tag{4.4.48}$$

On the other hand combining this last equality with the green formula and :

$$\hat\nu\mathcal{C}^{(1)}\partial_{\hat\nu} d_{\hat\nu}^{-2}P(x_\Gamma;\hat\nu)\big|_{\hat\nu=-1} = 0, \tag{4.4.49}$$

yields $\widehat{\operatorname{div}}\left(\hat\nu\mathcal{C}^{(1)}\partial_{\hat\nu} d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot) = \partial_{\hat\nu}\left(\hat\nu 2H(x_\Gamma)\partial_{\hat\nu} d_{\hat\nu}^{-2}P(x_\Gamma;\hat\nu)\right)$ in $\mathbb{C}[\hat\nu]$. Therefore reusing (4.4.49) leads to:

$$\mu\Big(\widehat{\operatorname{div}}\left(\hat\nu\mathcal{C}^{(1)}\partial_{\hat\nu} d_{\hat\nu}^{-2}P\right)(x_\Gamma;\cdot)\Big) = \int_{\hat\nu=-1}^{0} \partial_{\hat\nu}\left(\hat\nu\mathcal{C}^{(1)}\partial_{\hat\nu} d_{\hat\nu}^{-2}P(x_\Gamma;\hat\nu)\right)d\hat\nu = \cancel{\left[\hat\nu\mathcal{C}^{(1)}\partial_{\hat\nu} d_{\hat\nu}^{-2}P(x_\Gamma;\hat\nu)\right]_{\hat\nu=-1}^{\hat\nu=0}} = 0.$$

Combining this with (4.4.48) and (4.4.47) conclude the proof of (4.4.46).

Now let us prove that for all $i=1,2$:

$$A(x_\Gamma) := \int_{\hat Y_-} P(x_\Gamma;\hat\nu) w_i(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu - \langle Q(x_\Gamma;\cdot), w_i(x_\Gamma;\cdot)\rangle_{\hat Y_\infty} = 0. \tag{4.4.50}$$

Indeed, from (4.4.43), applying Corollary 2.5.7 (See Chapter 2) with $\phi = w_i(x_\Gamma;\cdot)$ yields:

$$A(x_\Gamma) = \int_{\hat Y_-} \hat\rho(x_\Gamma;\hat x,\hat\nu)\partial_{\hat\nu} d_{\hat\nu}^{-2}P(x_\Gamma;\hat\nu)\cdot\partial_{\hat\nu} w_i(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu \tag{4.4.51}$$

Let $\chi :]-1,\infty[\mapsto[0,1]$ be a C^∞ cut off function such that $\chi\equiv 1$ on $[-1,0]$ and $\phi\equiv 0$ on $]2,\infty[$. We recall that w_i satisfy the $\mathcal{P}_1^\infty$ property. Thus w_i satisfies (4.3.24), which leads to:

$$\forall\hat\nu>0,\ \int_{]0,1[^2\times\{\hat\nu\}} w_i(x_\Gamma;\hat x,\hat\nu)d\hat x = 0. \tag{4.4.52}$$

Moreover $\chi\cdot d_{\hat\nu}^{-2}P$ does not depend on $\hat x$. Combining this with (4.4.52) yields:

$$\int_{\hat Y_+} \partial_{\hat\nu}\big(\chi(\hat\nu) d_{\hat\nu}^{-2}P(x_\Gamma;\hat x,\hat\nu)\big)\cdot\partial_{\hat\nu} w_i(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu = 0.$$

Adding this with (4.4.51) yields:

$$\begin{aligned} A(x_\Gamma) &= \int_{\hat Y_\infty}\rho(x_\Gamma;\hat x,\hat\nu)\partial_{\hat\nu}\big(\chi(\hat\nu) d_{\hat\nu}^{-2}P(x_\Gamma;\hat\nu)\big)\cdot\partial_{\hat\nu} w_i(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu,\\ &= -\langle\left(\mathcal{T}_0 w_i\right)(x_\Gamma;\cdot), \chi\cdot d_{\hat\nu}^{-2}P(x_\Gamma;\cdot)\rangle_{\hat Y_\infty}. \end{aligned}$$

Combining this with the definition of w_i given in (4.3.18) yields:

$$A(x_\Gamma) = \langle \partial_{\hat{x}_i}\hat{\rho}(x_\Gamma;\cdot), \chi \cdot d_{\hat{\nu}}^{-2}P(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty} = \int_{\hat{Y}_\infty} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\chi(\hat{\nu})\cancel{\partial_{\hat{x}_i} d_{\hat{\nu}}^{-2}P(x_\Gamma;\hat{x},\hat{\nu})}d\hat{x}d\hat{\nu} = 0,$$

which conclude the proof of (4.4.50). Since for all $x_\Gamma \in$ the quantity $d_{\hat{\nu}}^{-2}P(x_\Gamma)$ only depend on $\hat{x}$ then the vector $\hat{\rho}\,\widehat{\nabla}\, d_{\hat{\nu}}^{-2}$ is co-linear to the normal n which leads to:

$$\mu\Big(\operatorname{div}_\Gamma\left(\hat{\rho}\,\widehat{\nabla}\, d_{\hat{\nu}}^{-2}P\right)(x_\Gamma;\cdot)\Big) = 0.$$

Combining this last equation with (4.4.46) and (4.4.50) yields (4.4.45). Finally adding (4.4.44) and (4.4.45) yields the stated results which ends the proof. $\square$

To state a new result of simplification of the quantity l_2^α, one need to introduce some quantities defined on the surface Γ. First we introduce $\mathbf{w} : \Gamma \mapsto \big(\mathbb{H}(\hat{Y}_\infty)\big)^3$ defined by:

$$\mathbf{w} := w_i e^i. \tag{4.4.53}$$

Then we define the function $W_\Gamma : \Gamma \mapsto \mathbb{H}(\hat{Y}_\infty)^\dagger$ by $W_\Gamma := \mathcal{T}_1\big(\mathbf{w}\cdot\nabla_\Gamma u_0\big)$. Finally we define $V_\Gamma : \Gamma \mapsto \mathbb{R}^3$ as the unique tangential field such that for all $i \in \{1,2\}$ and $x_\Gamma \in \Gamma$ we have:

$$(V_\Gamma(x_\Gamma), e_i(x_\Gamma)) = \langle W_\Gamma(x_\Gamma;\cdot), w^i(x_\Gamma;\cdot)\rangle_{\hat{Y}_\infty}.$$

Now we can state the following result:

Corollary 4.4.6. *For all $x_\Gamma \in \Gamma$ we have:*

$$\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) = -\operatorname{div}_\Gamma\left(V_\Gamma(x_\Gamma)\right).$$

Proof. Thanks to Proposition 4.3.2 we have $\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0 = w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)$ and then acording to the definition of W_Γ this becomes $\big(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_1\big)\gamma_\Gamma u_0 = W_\Gamma$. Thus applying Proposition 4.4.5 directly yields our result. $\square$

We define for $x_\Gamma \in \Gamma$ the tensor $\boldsymbol{M}_{1,0}^\rho$ if $x_\Gamma \in \Gamma_{\mathrm{M}}$ by the unique element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ such that for all $(i,j) \in \{1,2\}^2$:

$$\big(\boldsymbol{M}_{1,0}^\rho(x_\Gamma)\, e_i(x_\Gamma), e_j(x_\Gamma)\big) := \int_{\hat{Y}_\infty} \rho(x_\Gamma)\hat{\nu}(\mathcal{C}^{(1)}\hat{\nabla}w_i(x_\Gamma;\hat{x},\hat{\nu}), \hat{\nabla}w_j(x_\Gamma;\hat{x},\hat{\nu}))d\hat{x}d\hat{\nu}, \tag{4.4.54}$$

else if $x_\Gamma \notin \Gamma_{\mathrm{M}}$ this last quantity is defined by 0.

Finally, we introduce the "density of tensor field" $\widehat{\nabla}\,\boldsymbol{w}$ defined for $x_\Gamma \in \Gamma_{\mathrm{M}}$ by:

$$\forall i \in \{1,2\},\ \widehat{\nabla}\,\boldsymbol{w}(x_\Gamma;\cdot)e_i(x_\Gamma) := \widehat{\nabla}\, w_i(x_\Gamma;\cdot). \tag{4.4.55}$$

and then we can state the followings result:

Proposition 4.4.7. *We have for all $x_\Gamma \in \Gamma$:*

$$l_2^\alpha(x_\Gamma) = -\operatorname{div}_\Gamma\left(\left(\boldsymbol{M}_{1,0}^\rho(x_\Gamma) - \int_{\hat{Y}_\infty}(\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w}\star\boldsymbol{w})(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)\nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\right).$$

Proof. Thanks to Corollary 4.4.6 we get:

$$l_2^\alpha(x_\Gamma) = \mathrm{div}_\Gamma(Q_0 + Q_1 + Q_2)(x_\Gamma), \tag{4.4.56}$$

where $Q_1(x_\Gamma)$, $Q_1(x_\Gamma)$ and $Q_2(x_\Gamma)$ are the unique element of $T_{x_\Gamma}\Gamma$ such that for all i in $\{1,2\}$:

$$\begin{cases} \big(Q_0(x_\Gamma), e_i(x_\Gamma)\big) := \left\langle \widehat{\mathrm{div}}\left(\hat\rho\hat\nu\mathcal{C}^{(1)}\,\widehat\nabla(\mathbf{w}, \nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\cdot), w_i(x_\Gamma;\cdot)\right\rangle_{\hat Y_\infty}, \\ \big(Q_1(x_\Gamma), e_i(x_\Gamma)\big) := \left\langle \mathrm{div}_\Gamma\left(\hat\rho\,\widehat\nabla(\mathbf{w}, \nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\cdot), w_i(x_\Gamma;\cdot)\right\rangle_{\hat Y_\infty}, \\ \big(Q_2(x_\Gamma), e_i(x_\Gamma)\big) := \left\langle \widehat{\mathrm{div}}\left(\hat\rho\,\nabla_\Gamma(\mathbf{w}, \nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\cdot), w_i(x_\Gamma;\cdot)\right\rangle_{\hat Y_\infty}. \end{cases}$$

First prove that:

$$Q_0 = -\boldsymbol{M}^\rho_{1,0}\,\nabla_\Gamma\,u_0. \tag{4.4.57}$$

Indeed, according to the definition (4.4.55), we obtain:

$$\big(Q_0(x_\Gamma), e_i(x_\Gamma)\big) = \left\langle \widehat{\mathrm{div}}\left(\hat\rho\hat\nu\mathcal{C}^{(1)}\,\widehat\nabla\,w_i(x_\Gamma;\cdot)\right), w_j(x_\Gamma;\cdot)\right\rangle_{\hat Y_\infty}(e^j(x_\Gamma), \nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)).$$

According to the definition of the operator $\widehat{\mathrm{div}}$ given by (4.1.3), this becomes:

$$\big(Q_0(x_\Gamma), e_i(x_\Gamma)\big) = -\big(e^j(x_\Gamma), \nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\big)\int_{\hat Y_\infty}\left(\hat\rho\hat\nu(\mathcal{C}^{(1)}\,\widehat\nabla\,w_i, \widehat\nabla\,w_j)\right)(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu.$$

Thus thanks to (4.4.54), we obtain:

$$\big(Q_0(x_\Gamma), e_i(x_\Gamma)\big) = -\big((\boldsymbol{M}^\rho_{1,0}\cdot e_j, e_i)(e^j, \nabla_\Gamma\,u_0)\big)(x_\Gamma) = -\Big((\boldsymbol{M}^\rho_{1,0}\,\nabla_\Gamma\,\gamma_\Gamma u_0)(x_\Gamma), e_i(x_\Gamma)\Big),$$

which concludes the proof of (4.4.57).

Now, let us prove that for all $x_\Gamma \in \Gamma$ we have:

$$\mathrm{div}_\Gamma(Q_1 + Q_2)(x_\Gamma) = \mathrm{div}_\Gamma\left(\left(\int_{\hat Y_\infty}(\hat\rho\,\widehat\nabla\,\boldsymbol{w}\star\boldsymbol{w})(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu\right)\nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\right). \tag{4.4.58}$$

For convenience we introduce the density of operator $\mathcal{A} : \hat Y_\infty \mapsto \big(\mathcal{L}\big(H^3(\Gamma); L^2(\Gamma)\big)\big)$ given for $(\hat x, \hat\nu)$ by:

$$\mathcal{A}(\hat x,\hat\nu) := u \mapsto \mathrm{div}_\Gamma\left(\mathbf{w}(.;\hat x,\hat\nu)\,\mathrm{div}_\Gamma\left(\hat\rho\,\widehat\nabla(\mathbf{w}(.;\hat x,\hat\nu), \nabla_\Gamma\,u)\right)\right), \tag{4.4.59}$$

in order to get the following rewriting:

$$\mathrm{div}_\Gamma(Q_1) = \int_{\hat Y_\infty}\mathcal{A}(\hat x,\hat\nu)d\hat x d\hat\nu\cdot\gamma_\Gamma u_0. \tag{4.4.60}$$

This last quantity is well defined because thanks to (4.1.8) we have $\gamma_\Gamma u_0 \in H^3(\Gamma)$. Let us prove the following identity:

$$\mathrm{div}_\Gamma(Q_2) = \int_{\hat Y_\infty}\mathcal{A}^\dagger(\hat x,\hat\nu)d\hat x d\hat\nu\cdot\gamma_\Gamma u_0, \tag{4.4.61}$$

in the sense that for all $v \in H^3(\Gamma)$ we have:

$$\int_\Gamma \mathrm{div}_\Gamma\big(Q_2(x_\Gamma)\big)\overline{v}(x_\Gamma)dx_\Gamma = \int_\Gamma \gamma_\Gamma u_0(x_\Gamma)\overline{\int_{\hat{Y}_\infty} \mathcal{A}(\hat{x},\hat{\nu})\cdot v(x_\Gamma)d\hat{x}d\hat{\nu}}dx_\Gamma. \tag{4.4.62}$$

Let $v \in H^3(\Gamma)$. Thanks to (4.1.3) and Fubini's theorem, we have:

$$\begin{aligned}
-\int_\Gamma \big(Q_2(x_\Gamma),\nabla_\Gamma v(x_\Gamma)\big)dx_\Gamma &= \int_\Gamma \Big\langle \widehat{\mathrm{div}}\left(\hat{\rho}\,\nabla_\Gamma(\mathbf{w},\nabla_\Gamma(\gamma_\Gamma u_0))\right)(x_\Gamma;\cdot), w_i(x_\Gamma;\cdot)\Big\rangle_{\hat{Y}_\infty}\overline{G_i(x_\Gamma)}dx_\Gamma,\\
&= \int_{\Gamma\times\hat{Y}_\infty}\big(\hat{\rho}\,\nabla_\Gamma(\mathbf{w},\nabla_\Gamma(\gamma_\Gamma u_0)),\widehat{\nabla}\,w_i\big)(x_\Gamma;\hat{x},\hat{\nu})\overline{G_i(x_\Gamma)}dx_\Gamma d\hat{x}d\hat{\nu}.
\end{aligned}$$

where we defined for $x_\Gamma \in \Gamma$ $G_i(x_\Gamma) := \big(\nabla_\Gamma v(x_\Gamma), e^i(x_\Gamma)\big)$. Thanks to (4.1.3) and the surface Green formula for the ∇_Γ operator, this becomes

$$\begin{aligned}
-\int_\Gamma \big(Q_2(x_\Gamma),\nabla_\Gamma v(x_\Gamma)\big)dx_\Gamma &= -\int_{\Gamma\times\hat{Y}_\infty}\big(\mathbf{w},\nabla_\Gamma(\gamma_\Gamma u_0)\big)(x_\Gamma;\hat{x},\hat{\nu})\overline{\mathrm{div}_\Gamma\left(\hat{\rho}\,\widehat{\nabla}(G_i w_i)\right)}(x_\Gamma;\hat{x},\hat{\nu})dx_\Gamma d\hat{x}d\hat{\nu},\\
&= \int_{\Gamma\times\hat{Y}_\infty}\gamma_\Gamma u_0(x_\Gamma)\overline{\mathrm{div}_\Gamma\Big(\mathbf{w}\,\mathrm{div}_\Gamma\left(\hat{\rho}\,\widehat{\nabla}(G_i w_i)\right)\Big)}(x_\Gamma;\hat{x},\hat{\nu})dx_\Gamma d\hat{x}d\hat{\nu}.
\end{aligned}$$

Thanks to (4.4.59), this can rewritten as follow:

$$-\int_\Gamma \big(Q_2(x_\Gamma),\nabla_\Gamma v(x_\Gamma)\big)dx_\Gamma = \int_{\Gamma\times\hat{Y}_\infty}\gamma_\Gamma u_0(x_\Gamma)\overline{\mathcal{A}(\hat{x},\hat{\nu})v(x_\Gamma)}dx_\Gamma d\hat{x}d\hat{\nu}.$$

From this equality and the surface Green formula and the Fubini theorem we can easily conclude the proof of (4.4.62).

Adding (4.4.60) and (4.4.61) yields:

$$\mathrm{div}_\Gamma(Q_1+Q_2) = \int_{\hat{Y}_\infty} 2\cdot\mathrm{Re}\big(\mathcal{A}(\hat{x},\hat{\nu})\big)d\hat{x}d\hat{\nu}\cdot\gamma_\Gamma u_0.$$

According to Proposition 4.4.4, this last equality becomes (4.4.58). Combining (4.4.58) with (4.4.56) and (4.4.57) yields the stated result. □

4.4.4.3 The term $l_2^\beta(x_\Gamma)$

We introduce for convenience the following tensor field defined for $x_\Gamma \in \Gamma$ as follow

- If $x_\Gamma \in \Gamma_\mathrm{M}$, $\boldsymbol{M}^\rho_{1,3}(x_\Gamma)$ is the unique element of $T_{x_\Gamma}\Gamma$ such that for all $i \in \{1,2\}$:

$$\forall i \in \{1,2\},\ \boldsymbol{M}^\rho_{1,3}(x_\Gamma)\cdot e_i(x_\Gamma) := \int_{\hat{Y}_\infty}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\hat{\nu}\,\widehat{\nabla}\,w_i(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} \tag{4.4.63}$$

- If $x_\Gamma \notin \Gamma_\mathrm{M}$, $\boldsymbol{M}^\rho_{1,3}(x_\Gamma) = 0$.

Moreover, we introduce the scalar field $\mathcal{M}_1$ defined for $x_\Gamma \in \Gamma$ by:

$$\mathcal{M}_1(x_\Gamma) := \int_{\hat{Y}_-}\mathrm{div}_\Gamma\left(\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\boldsymbol{w}(x_\Gamma;\hat{x},\hat{\nu})\right)d\hat{x}d\hat{\nu}, \tag{4.4.64}$$

in order to state the following result:

Proposition 4.4.8. *We have for all $x_\Gamma \in \Gamma$:*

$$l_2^\beta(x_\Gamma) = \operatorname{div}_\Gamma\left(\left(\left(\mathcal{C}^{(1)}\boldsymbol{M}_{1,3}^\rho + (\mathcal{C}^{(1)}\boldsymbol{M}_{1,3}^\rho)^\dagger\right)(x_\Gamma) + \int_{\hat{Y}_-} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\mathbb{I}\star\boldsymbol{w}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)\nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\right),$$
$$\qquad + k^2\mathcal{M}_1(x_\Gamma)\gamma_\Gamma u_0(x_\Gamma).$$

Proof. We recall that for all $i \in \{1,2\}$, w_i satisfies the $\mathcal{P}_{m_\Gamma}^\infty$ property. Therefore by combining, (4.3.24) with the property of decreasing of the functions $(\phi_j)_{j\in\mathbb{Z}^2\setminus\{0\}}$, we can prove that for all $x_\Gamma \in \Gamma$:

$$\left(\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(2)}\hat{\nu}^2\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot),\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)\right) \in \left(\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\right)^2.$$

Thus, we have from the definition of the operator $\widehat{\operatorname{div}}$ and μ respectively given by (4.1.3) and (4.1.6) that:

$$\begin{cases} \mu\Big(\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(2)}\hat{\nu}^2\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big) = \langle\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(2)}\hat{\nu}^2\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty} = 0, \\ \mu\Big(\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big) = \langle\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot),1\rangle_{\hat{Y}_\infty} = 0. \end{cases}$$

Thus we have from the definition of the operator $\mathcal{T}_2$ given in (4.1.4) and Proposition 4.3.2 that:

$$\begin{aligned}
\mu\Big(\left(\mathcal{T}_2\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) =& \mu\Big(\left(\mathcal{T}_2\big(w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)\big)\right)(x_\Gamma;\cdot)\Big) \\
=& \cancel{\mu\Big(\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(2)}\hat{\nu}^2\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big)} \\
&+\mu\Big(\operatorname{div}_\Gamma\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big) \\
&+\cancel{\mu\Big(\widehat{\operatorname{div}}\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big)} \\
&+\mu\Big(\operatorname{div}_\Gamma\left(\hat{\rho}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big) \\
&+k^2\cdot\mu\Big(\left(\hat{\mu}w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)\Big),
\end{aligned}$$

Moreover, by using Proposition 2.5.15 and Proposition 2.5.16 (See Chapter 2), we can prove that the following distributions

$$\operatorname{div}_\Gamma\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot),\ \operatorname{div}_\Gamma\left(\hat{\rho}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot),$$

and $\left(\hat{\mu}w_i(e^i,\nabla_\Gamma u_0)\right)(x_\Gamma;\cdot)$ belong to the space $\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger$. Therefore according to the definition of the operator $\widehat{\operatorname{div}}$ and μ respectively given by (4.1.3) and (4.1.6), we have:

$$\begin{aligned}
\mu\Big(\left(\mathcal{T}_2\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) =& \int_{\hat{Y}_\infty}\operatorname{div}_\Gamma\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\widehat{\nabla}\,w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} \\
&+\int_{\hat{Y}_\infty}\operatorname{div}_\Gamma\left(\hat{\rho}\,\nabla_\Gamma\,w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} \\
&+k^2\cdot\int_{\hat{Y}_\infty}\left(\hat{\mu}w_i(e^i,\nabla_\Gamma\,\gamma_\Gamma u_0)\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}.
\end{aligned}$$

Thanks to the definition of the tensor field $\boldsymbol{M}^{\rho}_{1,3}$ given in (4.4.63), this becomes:

$$\begin{aligned}\mu\Big(\left(\mathcal{T}_2\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) &= \mathrm{div}_\Gamma\left(\mathcal{C}^{(1)}\boldsymbol{M}^{\rho}_{1,3}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)(x_\Gamma)\\ &\quad+\left(\int_{\hat{Y}_\infty}\mathcal{B}(\hat{x},\hat{\nu})\cdot\gamma_\Gamma u_0 d\hat{x}d\hat{\nu}\right)(x_\Gamma),\end{aligned}\tag{4.4.65}$$

where we defined the density of operator $\mathcal{B}:\hat{Y}_\infty\mapsto\left(\mathcal{L}\big(H^3(\Gamma);L^2(\Gamma)\big)\right)$ given for $(\hat{x},\hat{\nu})$ by:

$$\mathcal{B}(\hat{x},\hat{\nu}) := u\mapsto \mathrm{div}_\Gamma\left(\hat{\rho}(.;\hat{x},\hat{\nu})\,\nabla_\Gamma\left(\widehat{\nabla}(\mathbf{w}(.;\hat{x},\hat{\nu}),\nabla_\Gamma\,u)\right)\right)+k^2\cdot\hat{\mu}(\mathbf{w}(.;\hat{x},\hat{\nu}),\nabla_\Gamma\,u).$$

We emphasize that $\mathcal{B}(\hat{x},\hat{\nu})$ is well defined because 4.1.8 states that $\gamma_\Gamma u_0\in H^3(\Gamma)$. Thanks to Proposition 4.4.5 we have:

$$\begin{aligned}\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_2\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) &= -\,\mathrm{div}_\Gamma\left(e^i(x_\Gamma)\left\langle\widehat{\mathrm{div}}\left(\hat{\rho}\mathcal{C}^{(1)}\hat{\nu}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)(x_\Gamma;\cdot),w_i(x_\Gamma;\cdot)\right\rangle_{\hat{Y}_\infty}\right)\\ &\quad-\mathrm{div}_\Gamma\left(\int_{\hat{Y}_\infty}\mathbf{w}(x_\Gamma;\hat{x},\hat{\nu})\,\mathrm{div}_\Gamma\left(\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\,\nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\right)d\hat{x}d\hat{\nu}\right),\\ &\quad-k^2\,\mathrm{div}_\Gamma\left(\int_{\hat{Y}_\infty}\mathbf{w}(x_\Gamma;\hat{x},\hat{\nu})\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\gamma_\Gamma u_0 d\hat{x}d\hat{\nu}\right),\end{aligned}$$

and thanks to the definition of $\mathcal{B}$ and 4.1.3 this becomes

$$\begin{aligned}\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_2\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) &= \mathrm{div}_\Gamma\left(e^i(x_\Gamma)\int_{\hat{Y}_\infty}\left(\hat{\rho}(\mathcal{C}^{(1)}\hat{\nu}\,\nabla_\Gamma\,\gamma_\Gamma u_0,\widehat{\nabla}\,w_i)\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)\\ &\quad+\left(\int_{\hat{Y}_\infty}\mathcal{B}^\dagger(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\gamma_\Gamma u_0\right)(x_\Gamma),\end{aligned}$$

which leads combined with the definition of the tensor field $\boldsymbol{M}^{\rho}_{1,3}$ given in (4.4.63) to:

$$\begin{aligned}\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_2\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big) &= \mathrm{div}_\Gamma\left((\mathcal{C}^{(1)}\boldsymbol{M}^{\rho}_{1,3})^\dagger\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)(x_\Gamma)\\ &\quad+\left(\int_{\hat{Y}_\infty}\mathcal{B}^\dagger(\hat{x},\hat{\nu})\cdot\gamma_\Gamma u_0 d\hat{x}d\hat{\nu}\right)(x_\Gamma),\end{aligned}\tag{4.4.66}$$

On the other hand, thanks to Proposition 4.4.3 and Proposition 4.4.4 we have for all $(\hat{x},\hat{\nu})\in\hat{Y}_\infty$:

$$2\mathrm{Re}\mathcal{B}(\hat{x},\hat{\nu})\gamma_\Gamma u_0 = \mathrm{div}_\Gamma\left(\left(\hat{\rho}\mathbb{I}\star\boldsymbol{w}\right)(.;\hat{x},\hat{\nu})\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)-k^2\,\mathrm{div}_\Gamma(\hat{\mu}(.;(\hat{x},\hat{\nu})\,\mathbf{w}(.;(\hat{x},\hat{\nu})))\gamma_\Gamma u_0\tag{4.4.67}$$

Adding (4.4.65) and (4.4.66) yields that the function $x_\Gamma\mapsto\mu\Big(\left(\mathcal{T}_2\mathcal{T}_0^{-1}\mathcal{T}_1\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big)+\mu\Big(\left(\mathcal{T}_1\mathcal{T}_0^{-1}\mathcal{T}_2\gamma_\Gamma u_0\right)(x_\Gamma;\cdot)\Big)$ is given by:

$$\mathrm{div}_\Gamma\left(2\mathrm{Re}(\mathcal{C}^{(1)}\boldsymbol{M}^{\rho}_{1,3})\,\nabla_\Gamma\,\gamma_\Gamma u_0\right)+\int_{\hat{Y}_\infty}2\mathrm{Re}\big(\mathcal{B}^\dagger(\hat{x},\hat{\nu})\big)\gamma_\Gamma u_0 d\hat{x}d\hat{\nu},$$

and combining this with (4.4.67) yields the desired result. □

4.4.5 Final boundary conditions of u_2 and the operator $\mathcal{Z}_2$

To summarize Proposition 4.4.2, Proposition 4.4.7 and Proposition 4.4.8, we recall that for all x_Γ we have:

$$
\begin{aligned}
l_2^\alpha(x_\Gamma) &= -\operatorname{div}_\Gamma\left(\left(\boldsymbol{M}^\rho_{1,0}(x_\Gamma) - \int_{\hat{Y}_\infty}(\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w}\star\boldsymbol{w})(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)\nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\right),\\
l_2^\beta(x_\Gamma) &= \operatorname{div}_\Gamma\left(\left((\mathcal{C}^{(1)}\boldsymbol{M}^\rho_{1,3} + (\mathcal{C}^{(1)}\boldsymbol{M}^\rho_{1,3})^\dagger)(x_\Gamma) + \int_{\hat{Y}_-}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})\mathbb{I}\star\boldsymbol{w}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)\nabla_\Gamma\,\gamma_\Gamma u_0(x_\Gamma)\right),\\
&\quad + k^2\mathcal{M}_1(x_\Gamma)\gamma_\Gamma u_0(x_\Gamma),\\
l_2^\gamma &= \operatorname{div}_\Gamma\left(\overline{\rho}_1(H-R)\,\nabla_\Gamma\,\gamma_\Gamma u_0\right) + k^2\overline{\mu}_1 H\gamma_\Gamma u_0.
\end{aligned}
$$

We introduce the tensor field :

$$
\boldsymbol{\rho}^1_{eff} := \boldsymbol{M}^\rho_1 + \boldsymbol{N}^\rho_1 + \overline{\rho}_1(H-R),
$$

where we defined :

$$
\boldsymbol{M}^\rho_1 := -\boldsymbol{M}^\rho_{1,0} + \mathcal{C}^{(1)}\boldsymbol{M}^\rho_{1,3} + (\mathcal{C}^{(1)}\boldsymbol{M}^\rho_{1,3})^\dagger \quad \text{and} \quad \boldsymbol{N}^\rho_1 := \int_{\hat{Y}_\infty}\hat{\rho}(\cdot;\hat{x},\hat{\nu})\Big(\mathbb{I}_{\hat{\nu}<0}\mathbb{I} + \widehat{\nabla}\,\boldsymbol{w}\Big)\star\boldsymbol{w}(\cdot;\hat{x},\hat{\nu})\Big)d\hat{x}d\hat{\nu},
$$

because we have the following rewriting:

$$
l_2^\alpha + l_2^\beta + l_2^\gamma = \operatorname{div}_\Gamma\left(\boldsymbol{\rho}^1_{eff}\,\nabla_\Gamma\,\gamma_\Gamma u_0\right) + k^2\overline{\mu}_1 H\gamma_\Gamma u_0 + k^2\mathcal{M}_1(x_\Gamma)\gamma_\Gamma u_0(x_\Gamma). \tag{4.4.68}
$$

We recall that the operators $\star$, $\boldsymbol{w}$ and $\widehat{\nabla}\,\boldsymbol{w}$ are respectively defined in (4.4.37), (4.4.53) and (4.4.55). We recall that $\boldsymbol{M}^\rho_{1,0}$ and $\boldsymbol{M}^\rho_{1,3}$ are defined through to the solution $(w_i)_{1\le i\le 2}$ of cell problems (4.3.17) as follow:

- For $x_\Gamma \in \Gamma_{\mathrm{M}}$, these tensors are the unique elements of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ such that for all $i \in \{1,2\}$:

$$
\left\{
\begin{aligned}
\left(\boldsymbol{M}^\rho_{1,0}(x_\Gamma)\,e_i(x_\Gamma), e_j(x_\Gamma)\right) &:= \int_{\hat{Y}_\infty}\left(\hat{\rho}\hat{\nu}(\mathcal{C}^{(1)}\hat{\nabla}w_i, \hat{\nabla}w_j)\right)(x_\Gamma;\cdot)d\hat{x}d\hat{\nu},\\
\left(\boldsymbol{M}^\rho_{1,3}(x_\Gamma)\cdot e_i(x_\Gamma), e_j(x_\Gamma)\right) &:= \int_{\hat{Y}_\infty}\left(\hat{\rho}\hat{\nu}\,\widehat{\nabla}\,w_i, e_j\right)(x_\Gamma;\cdot)d\hat{x}d\hat{\nu},
\end{aligned}
\right.
$$

- For $x_\Gamma \notin \Gamma$, $\boldsymbol{M}^\rho_{1,0}(x_\Gamma) := 0$ and $\boldsymbol{M}^\rho_{1,3}(x_\Gamma) := 0$.

Finally, we recall that we defined for $x_\Gamma \in \Gamma$:

$$
\overline{\rho}_1(x_\Gamma) := \int_{\hat{Y}_-} 2\cdot\hat{\nu}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}, \quad \overline{\mu}_1(x_\Gamma) := \int_{\hat{Y}_-} 2\cdot\hat{\nu}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},
$$

and

$$
\mathcal{M}_1(x_\Gamma) := \int_{\hat{Y}_-}\operatorname{div}_\Gamma\left(\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\boldsymbol{w}(x_\Gamma;\hat{x},\hat{\nu})\right)d\hat{x}d\hat{\nu},
$$

Thus we can introduce the operator $\mathcal{Z}_2 : H^1 \mapsto H^{-1}(\Gamma)$ defined for u in $H^1(\Gamma)$ by:

$$
\mathcal{Z}_2 u := \operatorname{div}_\Gamma\left(\boldsymbol{\rho}^1_{eff}\,\nabla_\Gamma\,u\right) + k^2\cdot\left(H\overline{\mu}_1 - \mathcal{M}_1\right)u.
$$

Thanks to (4.4.68), we obtain the followings result:

Lemma 4.4.9. *The term u_2 is the unique solution of: Find $u_2 \in H^1(\Omega_0)$ such that for all $v \in H^1(\Omega_0)$:*

$$
a_0(u_2, v) = \langle \mathcal{Z}_1\gamma_\Gamma u_1, v\rangle_{\Gamma\times\{0\}} + \langle \mathcal{Z}_2\gamma_\Gamma u_0, v\rangle_{\Gamma\times\{0\}}.
$$

4.4.5.1 Case of symmetric cells

Definition 4.4.10. *We says that the cells of the thin coat are symmetric if for all $x_\Gamma \in \Gamma$ and $(\hat{x}, \hat{\nu}) \in \hat{Y}_\infty$ we have:*

$$\hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) = \hat{\rho}(x_\Gamma; -\hat{x}, \hat{\nu}) \quad \textit{and} \quad \hat{\mu}(x_\Gamma; \hat{x}, \hat{\nu}) = \hat{\mu}(x_\Gamma; -\hat{x}, \hat{\nu}).$$

Under this last condition the following property of functions $(w_i)_i$ holds:

Proposition 4.4.11. *If the cells of the thin coat are symmetric then for all $x_\Gamma \in \Gamma$ and $(\hat{x}, \hat{\nu}) \in \hat{Y}_\infty$ we have:*

$$\forall i \in \{1, 2\}, \; w_i(x_\Gamma; \hat{x}, \hat{\nu}) = -w_i(x_\Gamma; -\hat{x}, \hat{\nu}).$$

Proof. For all function f defined on $\Gamma \times \hat{Y}_\infty$ we define the function $S(f)$ for $(x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_\infty$

$$S(f)(x_\Gamma; \hat{x}, \hat{\nu}) := f(x_\Gamma; -\hat{x}, \hat{\nu}).$$

Thanks to this notation the symmetry property can be rewritten as:

$$S(\hat{\rho}) = \hat{\rho}.$$

Prove that for all $\phi \in \mathbb{H}(\hat{Y}_\infty)$ and $x_\Gamma \in \Gamma$ we have:

$$\int_{\hat{Y}_\infty} \big(\hat{\rho}(\widehat{\nabla}\, S(w_i), \widehat{\nabla}\, \phi)\big)(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \int_{\hat{Y}_\infty} \big(\hat{\rho}\partial_{\hat{x}_i}\big)(x_\Gamma; \hat{x}, \hat{\nu}) \phi d\hat{x} d\hat{\nu} \tag{4.4.69}$$

Indeed from $S(\hat{\rho}) = \hat{\rho}$ we have:

$$\hat{\rho}(\widehat{\nabla}\, S(w_i), \widehat{\nabla}\, \phi) = S\left(\hat{\rho}\big(\widehat{\nabla}\, w_i, \widehat{\nabla}\, \phi'\big)\right) \text{ with } S(\phi) := \phi' \in \mathbb{H}(\hat{Y}_\infty),$$

which leads to:

$$\int_{\hat{Y}_\infty} \big(\hat{\rho}(\widehat{\nabla}\, S(w_i), \widehat{\nabla}\, \phi)\big)(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \int_{\hat{Y}_\infty} \Big(\hat{\rho}\big(\widehat{\nabla}\, w_i, \widehat{\nabla}\, \phi'\big)\Big)(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}$$

Next, using $\mathcal{T}_0 w_i = \partial_{\hat{x}_i}\hat{\rho}$ yields that this last equality become:

$$\begin{aligned}\int_{\hat{Y}_\infty} \big(\hat{\rho}(\widehat{\nabla}\, S(w_i), \widehat{\nabla}\, \phi)\big)(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} &= -\int_{\hat{Y}_\infty} \hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{x_i}\phi'(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}, \\ &= \int_{\hat{Y}_\infty} S(\hat{\rho})(x_\Gamma; \hat{x}, \hat{\nu}) \partial_{x_i} S(\phi')(\hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}.\end{aligned}$$

Thus reusing $S(\hat{\rho}) = \hat{\rho}$ and $S(\phi') = \phi$ yields that this last equality become (4.4.69).

Therefore we have $\mathcal{T}_0 S(w_i) = -\partial_{x_i}\hat{\rho}$ and we clearly have $\delta_\Sigma(S(w_i)) = 0$ which leads to

$$S(w_i) - = (-\delta_\Sigma \otimes \delta_\Sigma + \mathcal{T}_0)^{-1} \partial_{x_i}\hat{\rho}$$

Thanks to (4.3.18) this becomes $S(w_i) = -w_i$ which conclude the proof. $\square$

Thanks to this result we get the following one:

Corollary 4.4.12. *If the cells of the thin coat are symmetric then we have (See* (4.4.63) *and* (4.4.64) *for definition of these quantities):*

$$\boldsymbol{N}_1^{\rho} = 0 \quad \textit{and} \quad \mathcal{M}_1 = 0.$$

Proof. We only give the proof of:

$$\int_{\hat{Y}_\infty} \big(\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w} \star \boldsymbol{w}\big)(;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} = 0,$$

because the arguments for the other quantities are the same. A sufficient condition is to prove that for all $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma_{\mathrm{M}} \times \hat{Y}_\infty$:

$$\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w} \star \boldsymbol{w}(x_\Gamma;\hat{x},\hat{\nu}) = -\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w} \star \boldsymbol{w}(x_\Gamma;-\hat{x},\hat{\nu}). \tag{4.4.70}$$

Indeed, both quantities appearing in this last equation vanishes if $x_\Gamma \in \Gamma \setminus \Gamma_{\mathrm{M}}$ and thanks to (4.4.37) we have for all $(i,j) \in \{1,2\}^2$ that:

$$(\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w} \star \boldsymbol{w}e^j, e^i) = \frac{1}{\sqrt{g}}\left[\partial_{x_k}(\sqrt{g}\alpha^{jk}\,\mathbf{w}^i - \sqrt{g}\,\mathbf{w}^k\,\alpha^{ji}) + \sqrt{g}\alpha^{ki}\partial_{x_k}\,\mathbf{w}^j - \mathbf{w}^i\,\partial_{x_k}(\sqrt{g}\alpha^{kj})\right], \tag{4.4.71}$$

where for all $(i,j) \in \{1,2\}^2$ $\alpha^{ij} := (\hat{\rho}\,\widehat{\nabla}\,\boldsymbol{w}e^j, e^i)$. We recall that ∂_{x_k} is defined in (4.4.36).

From Proposition 4.4.11 we get that for all $(i,j,k) \in \{1,2\}^3$ and $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma_{\mathrm{M}} \times \hat{Y}_\infty$:

$$\left(\alpha^{ij}\,\mathbf{w}^k\right)(x_\Gamma;\hat{x},\hat{\nu}) = -\left(\alpha^{ij}\,\mathbf{w}^k\right)(x_\Gamma;-\hat{x},\hat{\nu}).$$

Therefore we get for all $l \in \{1,2\}$:

$$\partial_{x_l}\left(\alpha^{ij}\,\mathbf{w}^k\right)(x_\Gamma;\hat{x},\hat{\nu}) = -\partial_{x_l}\left(\alpha^{ij}\,\mathbf{w}^k\right)(x_\Gamma;-\hat{x},\hat{\nu}).$$

Combining this last identity with (4.4.71) yields the sufficient condition (4.4.70) which ends the proof. $\square$

4.5 Construction of the effective boundary conditions and convergence

4.5.1 Formal construction of the effective boundary conditions

Thanks to Lemma 4.3.1, Lemma 4.3.8 and Lemma 4.4.9 we have on $\Gamma \times \{0\}$:

$$\partial_\nu u_i + \sum_{j=0}^{i} \mathcal{Z}^j \gamma_\Gamma u_{i-j} = 0.$$

Therefore, we introduce the operator $\mathcal{Z}_\delta^i := \sum_{j=0}^{i} \delta^j \mathcal{Z}^j$ because we formally get from this last equality that for all $i \in \{0,1,2\}$ the quantity $u_{i,\delta} := \sum_{j=0}^{i} \delta^j u_j$ formally satisfies on Γ:

$$\partial_\nu u_{i,\delta} + \mathcal{Z}_\delta^j \gamma_\Gamma u_{i,\delta} = O\left(\delta^{i+1}\right). \tag{4.5.72}$$

Moreover, we recall that this last quantity is an approximation of order i of $u_\delta \circ \mathcal{L}^{-1}$ which leads formally to for all $i \in \{0,1,2\}$ the exact solution u^δ satisfies on Γ:

$$\partial_\nu u^\delta + \mathcal{Z}_\delta^j \gamma_\Gamma u^\delta = O\left(\delta^{i+1}\right).$$

Thus we define for $i \in \{0,1,2\}$ the function $u_i^\delta : \Omega \mapsto \mathbb{C}$ as the unique solution of:

$$\Delta u_i^\delta + k^2 u_i^\delta = f \quad \text{and} \quad \partial_\nu u_i^\delta + \mathcal{Z}_\delta^i \gamma_\Gamma u_i^\delta = 0 \text{ on } \Gamma, \tag{4.5.73}$$

and u_i^δ satisfies the Sommerfeld radiation condition. We refer the reader to [25] to prove that this last problem is well posed in $H^1_{loc}(\Omega)$. We now prove some estimate which take the following form for all $i \in \{0,1,2\}$:

$$u^\delta = u_i^\delta + O\left(\delta^{i+1}\right), \tag{4.5.74}$$

and we proceed as follow:

1. We rewrite (4.5.73) as follow:

$$P_i^\delta\left(u_i^\delta \circ \mathcal{L}^{-1}\right) = f_{\Sigma_{\eta_0}},$$

   where $P_i^\delta : H^1(\Omega_0) \mapsto H^1(\Omega_0)^\dagger$ and $f^{\text{bis}}_{\Sigma_{\eta_0}} \in H^1(\Omega_0)^\dagger$ is defined for $v \in H^1(\Omega_0)$ by:

$$\langle f^{\text{bis}}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} := \langle f^{\text{bis}}_{\Sigma_{\eta_0}}, v\rangle_{\Gamma\times\{\eta_0\}}.$$

2. We prove existence of $C > 0$ independent of δ such that we have the following estimate:

$$\left\|P_i^\delta u_{i,\delta} - f^{\text{bis}}_{\Sigma_{\eta_0}}\right\|_{H^1(\Omega_0)^\dagger} \leq C\delta^{i+1},$$

   and this last property is called the "Consistence of the effective boundary conditions"

3. We prove existence of $C > 0$ independent of $\delta > 0$ such that the following estimate holds:

$$\left\|\left(P_i^\delta\right)^{-1}\right\|_{\mathcal{L}\left(H^1(\Omega_0)^\dagger, H^1(\Omega_0)\right)} \leq C.$$

   and this last property is called the "Stability of the effective boundary conditions".

4. We deduce the following estimate:

$$\left\|u_{i,\delta} - u_i^\delta \circ \mathcal{L}^{-1}\right\|_{H^1(\Omega_0)} \leq C\delta^{i+1} \tag{4.5.75}$$

5. We deduce (4.5.74) by combining (4.5.75) and Theorem 4.1.1 with the Triangle inequality.

4.5.2 Consistences of the effective boundary conditions

We give in this part a rigorous sense of the estimate (4.5.72). First, we chose to define for $i = 0, 1, 2$ the operator the operator as follow $\mathcal{Z}_\delta^i : H^1(\Gamma) \mapsto H^{-1}(\Gamma)$. Then, we introduce the space:

$$V_{\mathcal{Z}} := \left\{ u \in H^1(\Omega_0),\ \gamma_\Gamma u \in H^1(\Gamma) \right\},$$

and we provide this last space with the following norm:

$$\forall u \in V_{\mathcal{Z}},\ \|u\|_{V_{\mathcal{Z}}}^2 := \|u\|_{H^1(\Omega_0)}^2 + \|u\|_{H^1(\Gamma\times\{0\})}^2.k$$

Therefore for $i = 0, 1, 2$ we can define the sesquilinear form on $V_{\mathcal{Z}} \times V_{\mathcal{Z}}$ for $(u, v) \in V_{\mathcal{Z}}^2$ by:

$$a_\delta^i(u, v) := -\langle \mathcal{Z}_\delta^i \gamma_\Gamma u, \gamma_\Gamma v \rangle_{H^{-1}(\Gamma)-H^1(\Gamma)} + a_0(u, v). \tag{4.5.76}$$

Thanks to these last definitions we can state the following result:

Lemma 4.5.1. *For all $i = 0, 1, 2$, the function $u_{i,\delta}$ belongs to the space $V_{\mathcal{Z}}$ and there exists $C > 0$ such that for all $v \in V_{\mathcal{Z}}$ and $i = 0, 1, 2$ the following estimate holds:*

$$\left| a_\delta^i(u_{i,\delta}, v) - \langle f_{\Sigma_{\eta_0}}^{\text{bis}}, v \rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} \right| \le C\delta^{i+1} \|v\|_{H^1(\Omega_0)}.$$

Proof. From (4.1.8) we have:

$$\forall i \in \{0, 1, 2\},\ \gamma_\Gamma u_i \in H^{\frac{3}{2}}(\Gamma), \tag{4.5.77}$$

and using that $H^{\frac{3}{2}}(\Gamma) \subset H^1(\Gamma)$ yields $u_i \in V_{\mathcal{Z}}$.

We have seen that from (4.1.11) and (4.1.12), we have proved for all $i \in \{1, 2\}$ that $w_i \in C^{m_\Gamma}\Big(\Gamma; \mathbb{H}\big(\hat{Y}_\infty\big)\Big)$. Moreover we recall that $\boldsymbol{\rho}_{eff}^0$ and $\boldsymbol{\rho}_{eff}^1$, $\overline{\mu}_0$ are defined through these function. Thus we can easily prove that $\boldsymbol{\rho}_{eff}^0$, $\boldsymbol{\rho}_{eff}^1$, $\overline{\mu}_0$ and $\overline{\mu}_1$ are C^{m_Γ} functions. Therefore (4.5.77) leads to:

$$\forall (i, j) \in \{0, 1, 2\}^2,\ \mathcal{Z}_j \gamma_\Gamma u_i \in H^{-\frac{1}{2}}(\Gamma) \tag{4.5.78}$$

In the previous section we proved that for all $i \in \{0, 1, 2\}$:

$$a_0(u_i, v) = \delta_{0i} \langle f, v \rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} + \sum_{j=1}^{i} \langle \mathcal{Z}^j \gamma_\Gamma u_{i-j}, \gamma_\Gamma v \rangle_{H^{-1}(\Gamma)-H^1(\Gamma)}.$$

Thus we have:

$$\begin{aligned}
a_0(u_{i,\delta}, v) &= \langle f^{\rm bis}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} + \sum_{i'=0}^{i}\sum_{j=0}^{i'} \langle \mathcal{Z}^j \delta^{i'} \gamma_\Gamma u_{i'-j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)},\\
&= \langle f^{\rm bis}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} + \sum_{i'=0}^{i}\sum_{j=0}^{i'} \langle \mathcal{Z}^{i'-j} \delta^{i'} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)},\\
&= \langle f^{\rm bis}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} + \sum_{j=0}^{i}\sum_{i'=j}^{i} \langle \mathcal{Z}^{i'-j} \delta^{i'} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)},\\
&= \langle f^{\rm bis}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} + \sum_{j=0}^{i}\sum_{i'=0}^{i-j} \langle \mathcal{Z}^{i'} \delta^{i'+j} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)},\\
&= \langle f^{\rm bis}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} + \langle \mathcal{Z}^i_\delta \gamma_\Gamma u_{i,\delta}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)},\\
&+ \sum_{j=0}^{i}\sum_{i'=i-j+1}^{i} \langle \mathcal{Z}^{i'} \delta^{i'+j} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\left| a_0(u_{i,\delta}, v) - \langle f^{\rm bis}_{\Sigma_{\eta_0}}, v\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)}\right| &\le \sum_{j=0}^{i}\sum_{i'=i-j+1}^{i} \left|\langle \mathcal{Z}^{i'} \delta^{i'+j} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)}\right|,\\
&\le \delta^{i+1} \sum_{j=0}^{i}\sum_{i'=i-j+1}^{i} \left|\langle \mathcal{Z}^{i'} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)}\right|.
\end{aligned}$$

and thanks to (4.5.78) there exists $C > 0$ such that we have:

$$\sum_{j=0}^{i}\sum_{i'=i-j+1}^{i} \left|\langle \mathcal{Z}^{i'} \gamma_\Gamma u_{j}, \gamma_\Gamma v\rangle_{H^{-1}(\Gamma)-H^1(\Gamma)}\right| \le C\|v\|_{H^1\left(\Omega_0\right)},$$

which ends the proof. □

4.5.3 Stability

First prove the following intermediate result:

Proposition 4.5.2. *The tensor field $\boldsymbol{\rho}^0_{eff}$ is positive-definite in the sense that for all $x_\Gamma \in \Gamma$ and $\boldsymbol{u} \in T_{x_\Gamma}\Gamma$ we have:*

$$u(x_\Gamma) \neq 0 \Rightarrow (\boldsymbol{\rho}^0_{eff}(x_\Gamma)\boldsymbol{u}(x_\Gamma), \boldsymbol{u}(x_\Gamma)) > 0. \tag{4.5.79}$$

Moreover for δ small enough the tensor field:

$$\delta\boldsymbol{\rho}^0_{eff} + \delta^2\boldsymbol{\rho}^1_{eff},$$

is positive in the sense that for all $x_\Gamma \in \Gamma$ and $\boldsymbol{u} \in T_{x_\Gamma}\Gamma$ we have:

$$\Big((\delta\boldsymbol{\rho}^0_{eff}(x_\Gamma) + \delta^2\boldsymbol{\rho}^1_{eff}(x_\Gamma))\boldsymbol{u}, \boldsymbol{u}\Big) \ge 0. \tag{4.5.80}$$

Proof. **Proof of** (4.5.79) . Let $x_\Gamma \in \Gamma$ and $\boldsymbol{u} \in T_{x_\Gamma}\Gamma$. Assumes first that $x_\Gamma \notin \Gamma_{\mathrm{M}}$ or $u^i \,\widehat{\nabla}\, w_i(x_\Gamma) = 0$. In these two cases we have $\boldsymbol{M}_0^\rho(x_\Gamma) = 0$ which leads combined with (4.1.12) to:

$$(\boldsymbol{\rho}_{eff}^0(x_\Gamma)\boldsymbol{u}(x_\Gamma), \boldsymbol{u}(x_\Gamma)) = \overline{\rho}_0(x_\Gamma)|x|^2 > 0,$$

and then concludes the proof of (4.5.79).

Assume now that $x_\Gamma \in \Gamma_{\mathrm{M}}$ and $u^i \,\widehat{\nabla}\, w_i(x_\Gamma) \neq 0$. Thanks to (4.3.16), we have $\boldsymbol{u}{=}u^i e_i(x_\Gamma)$ with $u^i := \big(\boldsymbol{u}, e^i(x_\Gamma)\big)$. First prove that in this case the family $(u^i \,\widehat{\nabla}\, w_i(x_\Gamma;\cdot), \boldsymbol{u})$ is linearly independent. Indeed, in the contrary case the contrary case there would exists $\lambda \neq 0$ such that:

$$u^i \,\widehat{\nabla}\, w_i(x_\Gamma;\cdot) = \lambda \boldsymbol{u} \tag{4.5.81}$$

Since $w_i(x_\Gamma;\cdot)$ is $\hat{x}$ periodic, we have from Green formula for all $j = 1,2$ that:

$$\int_{\hat{Y}_-} (u^i \,\widehat{\nabla}\, w_i(x_\Gamma;\hat{x},\hat{\nu}), e^j(x_\Gamma))d\hat{x}d\hat{\nu} = 0.$$

Combining this with (4.5.81) yields that for all $j = 1,2$ we have $(\boldsymbol{u}, e^j(x_\Gamma)) = 0$. Therefore using that $\boldsymbol{u}$ belongs to $T_{x_\Gamma}\Gamma$ and that $(e^i(x_\Gamma))_i$ is a basis of $T_{x_\Gamma}\Gamma$ yields that $\boldsymbol{u} = 0$ which bring a contradiction and we now use this result. In order to apply Cauchy Schwartz inequality, we provide the space $L^2(\hat{Y}_\infty)^3$ with following dot product:

$$\forall (u,v) \in L^2(\hat{Y}_\infty)^3,\ (u,v)_{L^2(\hat{Y}_\infty)^3} := \int_{\hat{Y}_\infty} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})(u(\hat{x},\hat{\nu}), v(\hat{x},\hat{\nu}))d\hat{x}d\hat{\nu}. \tag{4.5.82}$$

Thanks to Proposition 4.3.4 we have:

$$\boldsymbol{M}_0^\rho(x_\Gamma)\boldsymbol{u} = \int_{\hat{Y}_-} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu}) \,\widehat{\nabla}\, w_i(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}u^i,$$

which leads combined with (4.5.82) to:

$$(\boldsymbol{M}_0^\rho(x_\Gamma)\boldsymbol{u}, \boldsymbol{u}) = (\widehat{\nabla}\, w_i(x_\Gamma;\cdot)u^i, \boldsymbol{u})_{\left(L^2(\hat{Y}_\infty)\right)^3}. \tag{4.5.83}$$

Since we proved that $\widehat{\nabla}\, w_i(x_\Gamma:\cdot)u^i$ and $\boldsymbol{u}$ are not co-linear then Cauchy Schwartz combined with (4.5.83) yields:

$$|(\boldsymbol{M}_0^\rho(x_\Gamma)\boldsymbol{u}, \boldsymbol{u})|^2 < \|\,\widehat{\nabla}\, w_i(x_\Gamma)u^i\|^2_{\left(L^2(\hat{Y}_\infty)\right)^3} \cdot \|\boldsymbol{u}\|^2. \tag{4.5.84}$$

Moreover, according to the definition of $\boldsymbol{M}_0^\rho(x_\Gamma)$ and $\overline{\rho}_0(x_\Gamma)$ respectively given by (4.3.22) and (4.3.25), we have the two following rewriting:

$$\begin{cases} \|\boldsymbol{u}\|^2_{\left(L^2(\hat{Y}_\infty)\right)^3} = \displaystyle\int_{\hat{Y}_-} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})|\boldsymbol{u}|^2 d\hat{x}d\hat{\nu} = \overline{\rho}_0(x_\Gamma)|\boldsymbol{u}|^2, \\ \left\|\widehat{\nabla}\, w_i(x_\Gamma)u^i\right\|^2_{\left(L^2(\hat{Y}_\infty)\right)^3} = \displaystyle\int_{\hat{Y}_-} \hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})|\,\widehat{\nabla}\, w_i(x_\Gamma;\hat{x},\hat{\nu})u^i|^2 d\hat{x}d\hat{\nu} \leq (\boldsymbol{M}_0^\rho(x_\Gamma)\boldsymbol{u}, \boldsymbol{u}). \end{cases}$$

Therefore combining this two last inequalities with (4.5.84) leads to $(\boldsymbol{M}_0^\rho(x_\Gamma)\boldsymbol{u}, \boldsymbol{u}) < \overline{\rho}_0(x_\Gamma)|\boldsymbol{u}|^2$ and combining this with the definition of $\boldsymbol{\rho}_{eff}^0$ given in (4.3.29) yields:

$$0 < \overline{\rho}_0(x_\Gamma)|\boldsymbol{u}|^2 - (\boldsymbol{M}_0^\rho(x_\Gamma)\boldsymbol{u}, \boldsymbol{u}) = \big(\boldsymbol{\rho}_{eff}^0(x_\Gamma)\boldsymbol{u}, \boldsymbol{u}\big).$$

which concludes the proof of (4.5.79).
Proof of (4.5.80): Thanks to (4.5.79), we can deduce that $\lambda_{\min}(\boldsymbol{\rho}^0) > 0$. Thus by using the compactness of Γ and the continuity of $\lambda_{\min}(\boldsymbol{\rho}^0)$ (see proof of Lemma 4.5.1) we have:

$$r_+ := \inf\left(\lambda_{\min}(\boldsymbol{\rho}^0_{eff})\right) > 0.$$

Moreover since $\lambda_{\max}(\boldsymbol{\rho}^1_{eff})$ is a smooth function on Γ (see proof of Lemma 4.5.1), we have:

$$r_- := \sup\left(\lambda_{\max}(\boldsymbol{\rho}^1_{eff})\right) < \infty.$$

Therefore for $\delta \leq r_+/(2r_-)$ we have:

$$\lambda_{\min}(\delta\boldsymbol{\rho}^0_{eff} + \delta^2\boldsymbol{\rho}^1_{eff}) \geq \delta r_+ - \delta^2 r_- \geq \frac{r_+}{2} > 0,$$

which concludes the proof of (4.5.80). □

Corollary 4.5.3. *There exist a compact operator $T_k : H^1(\Omega_0) \mapsto H^1(\Omega_0)^\dagger$ such that for all $u \in V_{\mathcal{Z}}$, $\delta > 0$ and $i = 0, 1, 2$ the following estimate holds:*

$$|a^i_\delta(u,u)| \geq \|u\|^2_{H^1(\Omega_0)} - \left|\langle T_k u, u\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)}\right|. \tag{4.5.85}$$

Proof. For this proof $\langle\cdot,\cdot\rangle$ is the dual product $\langle\cdot,\cdot\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)}$. Let $u \in V_{\mathcal{Z}}$ then we have:

$$a^i_\delta(u,u) = \langle (C + T_k)u, u\rangle, \tag{4.5.86}$$

where $C : H^1(\Omega_0) \mapsto H^1(\Omega_0)^\dagger$ and $T_k : H^1(\Omega_0) \mapsto H^1(\Omega_0)^\dagger$ are the only linear operators such that for all $(u,v) \in H^1(\Omega_0)^2$ we have:

$$\begin{cases} \langle Qu, v\rangle := \displaystyle\sum_{j=1}^{i} \delta^j (\boldsymbol{\rho}^{j-1}_{eff}\, \nabla_\Gamma u, \nabla_\Gamma v) + \int_{\Omega_0} ((\mathcal{C}\,\nabla_{\mathcal{L}} u, \nabla_{\mathcal{L}} v) + u\overline{v})\, d\Gamma d\nu + \langle \mathrm{DtN}^{k=i_{\mathcal{L}}}\, u, v\rangle_{\Gamma\times\{\eta_0\}}, \\ \langle T_k u, v\rangle := -\displaystyle\int_{\Omega_0} (1 + k^2\,\mathrm{C}) u\overline{v} d\Omega_0 + k^2 \cdot \sum_{j=1}^{i} \delta^i (\overline{\mu}_{i-1}\gamma_\Gamma u, \gamma_\Gamma v)_{L^2(\Gamma)} + \langle (\mathrm{DtN}_{\mathcal{L}} - \mathrm{DtN}^{k=i}_{\mathcal{L}})u, v\rangle_{\Gamma\times\{\eta_0\}}, \end{cases}$$

where $\mathrm{DtN}^{k=i}_{\mathcal{L}} : H^{\frac{1}{2}}(\Gamma\times\{\eta_0\}) \mapsto H^{-\frac{1}{2}}(\Gamma\times\{\eta_0\})$ is defined for $(u,v) \in H^{\frac{1}{2}}(\Gamma\times\{\eta_0\})^2$ by:

$$\left\langle \mathrm{DtN}^{k=i}_{\mathcal{L}}\, u, v\right\rangle_{\Gamma\times\{\eta_0\}} := \left\langle \mathrm{DtN}^{k=i}\, u\circ\mathcal{L}, v\circ\mathcal{L}\right\rangle_{\Sigma_{\eta_0}},$$

and $\mathrm{DtN}^{k=i} : H^{\frac{1}{2}}(\Sigma_0) \mapsto H^{-\frac{1}{2}}(\Sigma_0)$ is the Dirichlet to Neumann map on Σ_0 associated to the wave-number i.

Then we now prove the following coercivity property of the operator Q:

$$\langle Qu, u\rangle_{H^1(\Omega_0)^\dagger - H^1(\Omega_0)} \geq \|u\|^2_{H^1(\Omega_0)}. \tag{4.5.87}$$

Indeed, from the positivity of $\mathrm{DtN}^{k=i}$, one can show that for all $u \in H^{\frac{1}{2}}(\Gamma\times\{\eta_0\})$ we have:

$$\left\langle \mathrm{DtN}^{k=i}_{\mathcal{L}}\, u, u\right\rangle_{\Gamma\times\{\eta_0\}} \geq 0. \tag{4.5.88}$$

Thanks to Proposition 4.5.2 we have that $\sum_{j=1}^{i} \delta^j (\boldsymbol{\rho}_{eff}^{j-1} \nabla_\Gamma u, \nabla_\Gamma u) \geq 0$. Combining this with (4.5.88) concludes the proof of (4.5.87).

Now let us prove the compactness of the operator T_k. Indeed the operator $\mathrm{DtN} - \mathrm{DtN}^{k=1}$ is compact (Seed [60, Theorem 2.6.4], [51, appendix] and [25, Proposition 3.4]). Morover, thanks to Rellich lemma, the linear operator associated to the sesquilinear form defined for $(u, v) \in H^1\big(\Omega_0\big)^2$ by:

$$-\int_{\Omega_0} (1 + k^2\,\mathrm{C}) u\overline{v} d\Omega_0 + k^2 \cdot \sum_{j=1}^{i} \delta^i (\overline{\mu}_{i-1} u, v)_{L^2(\Gamma)},$$

is compact which concludes the proof of the compactness of T_k.

The estimate (4.5.85) is a direct consequence of (4.5.87) and (4.5.86) which concludes the proof. □

Lemma 4.5.4. *The effective boundary conditions are stable in the sense that there exists $C > 0$ such that for all $u \in V_{\mathcal{Z}}$ the following estimate holds:*

$$\|u\| \leq C \sup_{\|\phi\|_{H^1(\Omega_0)}=1} |a_\delta^i(u, \phi)|.$$

Proof. Let us prove this result by contradiction. Let u_δ such that

$$\|u_\delta\|_{H^1(\Omega_0)} = 1 \quad \text{and} \quad \lim_{\delta \to 0} \sup_{\|\phi\|_{H^1(\Omega_0)}=1} a_\delta^i(u_\delta, \phi) = 0. \tag{4.5.89}$$

Therefore the sequence u_δ is bounded in $H^1(\Omega_0)$ and then up to a sub-sequence there exists u_0 such that u_δ weakly converge to u_0. First prove that for all $\phi \in C^\infty(\overline{\Omega_0})$ we have:

$$a_0(u, \phi) = 0, \tag{4.5.90}$$

and using the fact that this last variational formulation is well posed and using density of the space $C^\infty(\Omega_0)$ into $H^1(\Omega_0)$ will deduce that $u_0 = 0$.

Indeed thanks to (4.5.89) and the definition (4.5.76) we have:

$$\limsup_{\delta \to 0} a_0(u_\delta, \phi) \leq \liminf_{\delta \to 0} \langle \mathcal{Z}_\delta^i \gamma_\Gamma u_\delta, \gamma_\Gamma \phi \rangle_{H^{-1}(\Gamma)-H^1(\Gamma)} = \liminf_{\delta \to 0} \langle \gamma_\Gamma u_\delta, \mathcal{Z}_\delta^i \gamma_\Gamma \phi \rangle_{H^{-1}(\Gamma)-H^1(\Gamma)}.$$

Moreover the function ϕ is regular on $\Gamma \times \{0\}$ and so thanks to the regularities of coefficients which appears on the operators $\mathcal{Z}_1$ and $\mathcal{Z}_2$ we have:

$$\liminf_{\delta \to 0} \langle \mathcal{Z}_\delta^i \gamma_\Gamma u_\delta, \gamma_\Gamma \phi \rangle_{H^{-1}(\Gamma)-H^1(\Gamma)} \leq \liminf_{\delta \to 0} \sum_{j=1}^{i} \delta^j \|u_\delta\|_{L^2(\Gamma \times \{0\})} \|\mathcal{Z}_i \phi\|_{L^2(\Gamma \times \{0\})} = 0.$$

Therefore we success to prove that u_δ weakly converge to 0.

Then we now use this weak convergence to prove a contradiction with (4.5.89). Indeed the compactness of the operator T_k implies that $T_k u_\delta \to 0$ strongly congerve to zero. Then according to Corollary 4.5.3, (4.5.89) leads to the strong convergence of u^δ to zero. This contradict (4.5.89). □

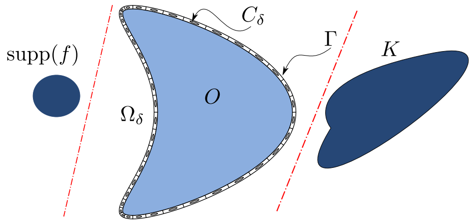


Figure 4.4: Illustration of the Final Convergence Theorem

4.5.4 Error estimate

Now we back to the initial geometry Ω^δ and define for $i \in \{0,1,2\}$ the function $u_i^\delta : \Omega \mapsto \mathbb{C}$ by the unique solution of:

$$\Delta u_i^\delta + k^2 u_i^\delta = f \quad \text{and} \quad \partial_\nu u_i^\delta = \mathcal{Z}_\delta^i u_i^\delta \text{ on } \Gamma,$$

and u_i^δ satisfies the Sommerfeld radiation condition. We refer the reader to [25] to prove that this last problem is well posed in $H^1_{loc}(\Omega)$. Thanks to all this work we can state the final result of this work.

Theorem 4.5.5 (Final Convergence Theorem). *For all $i \in \{0,1,2\}$, if $m_\Gamma \le 6+i$ and $\operatorname{supp}(f) \cap \Gamma = \emptyset$ then for all K open bounded subset of Ω such that $\overline{K} \cap \Gamma = \emptyset$ there exists C such that for all $\delta > 0$ we have:*

$$\|u^\delta - u_i^\delta\|_{H^1(K)} \le C\delta^{i+1}. \tag{4.5.91}$$

We refer the reader to Figure 4.4 for an illustration of assumptions of this last theorem.

Proof. With the same way of the proof Proposition 1.3.2 (See Chapter 1), we show that:

$$u_i^\delta = \tilde{u}_{i,\delta} \circ \mathcal{L}, \tag{4.5.92}$$

where $\tilde{u}_{i,\delta}$ is the unique solution of: Find $\tilde{u}_{i,\delta} \in V_\mathcal{Z}$ such that for all $v \in V_\mathcal{Z}$ we have:

$$a_\delta^i(u,\phi)(\tilde{u}_{i,\delta}, v) = \langle f_{\Sigma_{\eta_0}}, v\rangle.$$

Hereafter for this proof $C > 0$ is a generic constant independent of δ. Thanks to Lemma 4.5.1 we have:

$$\sup_{\phi \in V_\mathcal{Z}\setminus\{0\}} \frac{|a_\delta^i(u_{i,\delta} - \tilde{u}_{i,\delta}, \phi)|}{\|\phi\|_{H^1\left(\Omega_0\right)}} \le C\delta^{i+1}.$$

Therefore using Lemma 4.5.4 yields:

$$\|u_{i,\delta} - \tilde{u}_{i,\delta}\|_{H^1\left(\Omega_0\right)} \le C\delta^{i+1}. \tag{4.5.93}$$

Let $K \subset \Omega_\delta$ be a bounded open subset of Ω such that $\Gamma \cap \overline{K} = \emptyset$. Therefore from $\Gamma \cap \overline{K} = \emptyset$ and the compactness of Γ and $\overline{K}$ we have:

$$c := \frac{\operatorname{dist}(\Gamma, \overline{K})}{2} > 0.$$

Thus by applying Theorem 4.1.1, we have:

$$\|u_\delta - u_{i,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)} \leq C\delta^{i+1}.$$

Moreover combining this last estimate with (4.5.93) yields:

$$\|u_\delta - \tilde{u}_{i,\delta}\|_{H^1(\Gamma\times]c,\eta_0[)} \leq C\delta^{i+1},$$

Combining this with $u^\delta = u_\delta \circ \mathcal{L}$, (4.5.92) and [57, Theorem 3.20] yields:

$$\|u^\delta - u_i^\delta\|_{H^1(\mathcal{L}^{-1}(\Gamma\times]c,\eta_0[))} \leq C\delta^{i+1}. \tag{4.5.94}$$

Moreover we recall that $K \cap C_{\delta,\eta_0} \subset \mathcal{L}^{-1}(\Gamma\times]c,\eta_0[)$, which leads to:

$$\|u^\delta - u_i^\delta\|_{H^1(K\cap C_{\delta,\eta_0})} \leq C\delta^{i+1}. \tag{4.5.95}$$

Let us prove now:

$$\exists C > 0,\ \forall\delta > 0,\ \|u^\delta - u_i^\delta\|_{H^1(K\setminus\overline{C_{\delta,\eta_0}})} \leq C\delta^{i+1}. \tag{4.5.96}$$

Indeed we introduce the open bounded set $\tilde{O} := C_{\delta,\eta_0} \cup O$.

Thanks to Proposition 1.3.1 (See Chapter 1),

$$\partial\tilde{O} = \mathcal{L}^{-1}(\Gamma \times \{\eta_0\}). \tag{4.5.97}$$

From the regularity of the map $\mathcal{L}$ and Γ we deduce from (4.5.97) that $\tilde{O}$ is a Lipschitz domain. Therefore we can apply classical theory of scattering for Helmholtz equation which leads to:

$$\|u^\delta - u_i^\delta\|_{H^1(K\setminus\overline{\tilde{O}})} \leq C\|u^\delta - u_i^\delta\|_{H^{\frac{1}{2}}(\partial\tilde{O})}. \tag{4.5.98}$$

Moreover from (4.5.97) we have $\partial\tilde{O} \subset \partial\left(\mathcal{L}^{-1}(\Gamma\times]c,\eta_0[)\right)$ and combining with (4.5.94) yields:

$$\|u^\delta - u_i^\delta\|_{H^{\frac{1}{2}}(\partial\tilde{O})} \leq C\delta^{i+1}. \tag{4.5.99}$$

Since $K \cap \overline{O} = \emptyset$ we have $K \setminus \overline{\tilde{O}} = K \setminus \overline{C_{\delta,\eta_0}}$. Therefore combining (4.5.98) and (4.5.99) yields the desired result (4.5.96).

Adding (4.5.95) and (4.5.96) yields the desired result which ends the proof. □

Chapter 5

Numerical approximation of the approximate solution

This chapter contains two steps. We first, give a numerical procedure to compute a numerical approximation of the solution approximate model in the two dimensional case. That is the object of the section 5.3. Finally we will prove error estimates independent of the small parameter δ that only depends one the mesh size h. That is the object of section 5.4 and Theorem 5.4.2. We will see that it has an advantage compared to the exact model.

5.1 Two dimensional configuration

Hereafter, the dimension of our problem is two. We assume that our obstacle O is included on the ball $\mathbb{B}^{1/3} := \{p \in \mathbb{R}^2,\ |p| < 1/3\}$. For the sequel our domain is $\Omega := \mathbb{B}^{\frac{1}{3}} \setminus \overline{O}$. (See Figure 5.1) We assume that the boundary Γ is a parametric curve in the sense that there exists a function: $P : [0,1] \mapsto \mathbb{R}^2$ with $P(0) = P(1)$ such that:

$$\Gamma = \{P(t),\ t \in [0,1]\},$$

and this last function is supposed to be injective on $[0,1[$. To ensure that Γ is a C^∞ manifold, we assume that P is a C^∞ function and the velocity $\frac{dP}{dt}$ does not vanishe on $[0,1]$. The function ψ_Γ is defined by $P^{-1} : \Gamma \mapsto [0,1[$. The coefficient $\hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu})$ and

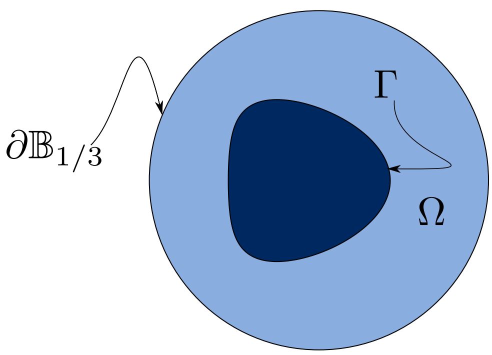


Figure 5.1: Illustration of the geometry

$\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})$ are supposed independent of x_Γ and we will write:

$$\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu}) = \hat{\rho}(\hat{x},\hat{\nu}) \quad \text{and} \quad \hat{\mu}(x_\Gamma;\hat{x},\hat{\nu}) = \hat{\mu}(\hat{x},\hat{\nu}).$$

We assume that our cell is symmetric in the sense that for all $(\hat{x},\hat{\nu}) \in \Omega :=]0,1[\times]-1,\infty[$ we have:

$$\hat{\rho}(\hat{x},\hat{\nu}) = \hat{\rho}(-\hat{x},\hat{\nu}) \quad \text{and} \quad \hat{\mu}(\hat{x},\hat{\nu}) = \hat{\mu}(-\hat{x},\hat{\nu}).$$

5.2 Reminder of the approximated problem

We recall that for all $i \in \{0,1,2\}$, the function u_i^δ is defined by the unique solution of: Find $u_i^\delta \in$:

$$V := \left\{ u \in H^1(\Omega), \int_\Gamma \left(|u|^2 + |\nabla_\Gamma u|^2\right) d\Gamma + \int_\Omega \left(|u|^2 + |\nabla u|^2\right) d\Omega < \infty \text{ and } u = 0 \text{ on } \partial\mathbb{B}^1 \right\},$$

such that for all $v \in V$ we have: $a_i^\delta(u,v) = (f,v)_{L^2(\Omega)}$. The sesquilinear form a_1^δ and a_2^δ are defined by:

$$a_1^\delta := a_0 + \delta a_1 \quad \text{and} \quad a_2^\delta := a_0 + \delta a_1 + \delta^2 a_2,$$

where a_0, a_1 and a_2 are defined for $(u,v) \in V \times V$ by:

$$a_0(u,v) := \int_\Omega \left(\nabla u \cdot \nabla v - k^2 uv\right) d\Omega \quad \text{and} \quad a_i(u,v) := \int_\Gamma \left(\boldsymbol{\rho}_{eff}^{i-1} \nabla_\Gamma u \cdot \nabla_\Gamma v - k^2 \overline{\mu}_{i-1} uv\right) d\Gamma\,, i = 1,2.$$

Here we define for all $x_\Gamma \in \Gamma$:

$$\boldsymbol{\rho}_{eff}^0(x_\Gamma) := \overline{\rho}_0 - \boldsymbol{M}_0^\rho\big(g(x_\Gamma)\big) \quad \text{and} \quad \boldsymbol{\rho}_{eff}^0 := -c(x_\Gamma)\overline{\rho}_1 + c(x_\Gamma)\boldsymbol{M}_1^\rho\big(g(x_\Gamma)\big)$$

where $c(x_\Gamma)$ is the curvature of Γ at the point x_Γ,

$$\overline{\rho}_i := \int_{\hat{Y}_-} i\hat{\nu}^{i-1}\hat{\rho}(\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} \quad \text{and} \quad \overline{\mu}_i := \int_{\hat{Y}_-} i\hat{\nu}^{i-1}\hat{\rho}(\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu},$$

$\hat{Y}_\infty :=]0,1[\times]-1,\infty[$ and $\hat{Y}_- :=]0,1[\times]-1,0[$ (see Figure 5.2) and $g : \Gamma \mapsto \mathbb{R}_+^*$ is given for x_Γ by

$$g(x_\Gamma) := \left|\frac{dP}{dt}(t)\right|^{-2} \quad \text{with} \quad t := P^{-1}(x_\Gamma).$$

The functions $\boldsymbol{M}_0^\rho, \boldsymbol{M}_1^\rho : \mathbb{R}_+^* \mapsto \mathbb{R}$ are computed through solutions w of "cell problems". These functions are defined for $g \in \mathbb{R}_+^*$ by:

$$\boldsymbol{M}_0^\rho(g) := g \cdot \int_{\hat{Y}_\infty} \hat{\rho}(\hat{x},\hat{\nu}) \left(g\big|\partial_{\hat{x}} w\big(g;\hat{x},\hat{\nu}\big)\big|^2 + \big|\partial_{\hat{\nu}} w\big(g;\hat{x},\hat{\nu}\big)\big|^2 \right) d\hat{x} d\hat{\nu},$$

and:

$$\boldsymbol{M}_1^\rho(g) := g \cdot \int_{\hat{Y}_\infty} \hat{\rho}(\hat{x},\hat{\nu}) \left(-g\big|\partial_{\hat{x}} w\big(g;\hat{x},\hat{\nu}\big)\big|^2 + \big|\partial_{\hat{\nu}} w\big(g;\hat{x},\hat{\nu}\big)\big|^2 \right) d\hat{x} d\hat{\nu} + 2g \int_{\hat{Y}_\infty} \hat{\rho}(\hat{x},\hat{\nu}) \hat{\nu} \partial_{\hat{x}} w\big(g;\hat{x},\hat{\nu}\big) d\hat{x} d\hat{\nu}.$$

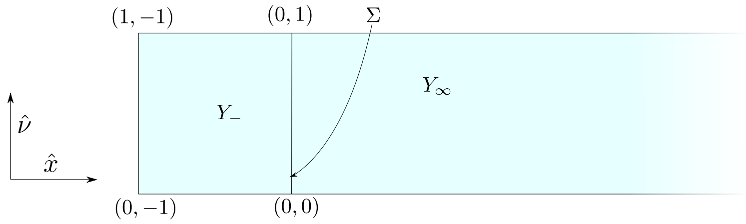


Figure 5.2: Illustration of the infinite strip

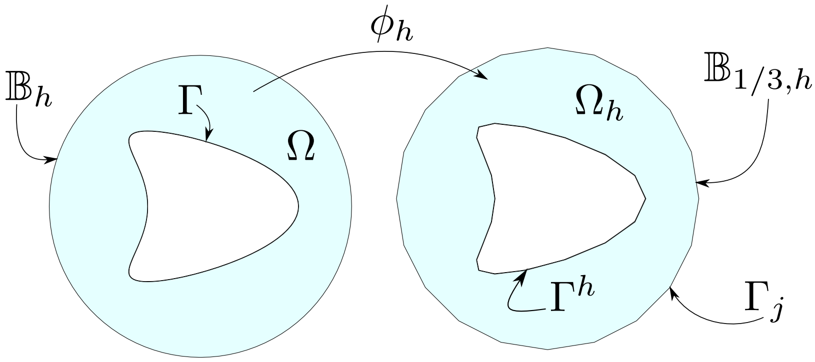


Figure 5.3: Illustration of the domain Ω_h and the map ϕ_h

Finally the function w is defined for $g > 0$ by $w(g;\cdot)$ which is the unique solution of (up to a constant): Find $w(g;\cdot) \in \mathbb{H}\big(\hat{Y}_\infty\big)$ such that for all $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ we have:

$$\int_{\hat{Y}_\infty} \hat{\rho}(\hat{x},\hat{\nu})\,\big(g\partial_{\hat{x}} w\big(g;\hat{x},\hat{\nu}\big)\partial_{\hat{x}} v(\hat{x},\hat{\nu}) + \partial_{\hat{\nu}} w\big(g;\hat{x},\hat{\nu}\big)\partial_{\hat{\nu}} v(\hat{x},\hat{\nu})\big)\,d\hat{x}d\hat{\nu} = \int_{\hat{Y}_\infty} \hat{\rho}(\hat{x},\hat{\nu})\partial_{\hat{x}} v(\hat{x},\hat{\nu}).d\hat{x}d\hat{\nu}$$

In this formulation we recall that

$$\mathbb{H}\big(\hat{Y}_\infty\big) := \Big\{u \in L^2_{\text{loc}}(\hat{Y}_\infty), \nabla u \in L^2(\hat{Y}_\infty), u \text{ is one periodic on the variable } \hat{x}\Big\}.$$

5.3 Construction of an approximation $u^{\boldsymbol{h}}_{i,\delta}$ of the exact function u^δ_i

Let $h > 0$ be a small number. Let Γ_h and $\partial\mathbb{B}_{1/3,h}$ be the triangulations of the surfaces Γ and $\mathbb{B}_{1/3}$ such that for all $j \in [0, 1/h] \cap \mathbb{N}$ we have

$$\Gamma_j := P(jh) \in \Gamma \quad \text{and} \quad 1/3\big(\cos(2\pi jh), \sin(2\pi jh)\big) \in \partial\mathbb{B}_{1/3},$$

and Ω_h be the polygonal open set such that $\partial\Omega_h = \Gamma_h \cup \partial\mathbb{B}_{1/3,h}$ (See Figure 5.3).

Let T_h be triangulation of the domain Ω_h and V_h the space of $P1$ function of the mesh T_h. We construct here an approximation $u^\delta_{i,\boldsymbol{h}} : \Omega_h \mapsto \mathbb{R}$ of the function u^δ_i. This function depends of a vector of small parameter $\boldsymbol{h} := (h, \hat{h}, 1/L, \Delta T) \mapsto 0$ where:

- h is the is the maximum of size of triangles of the mesh T_h of the domain Ω_h

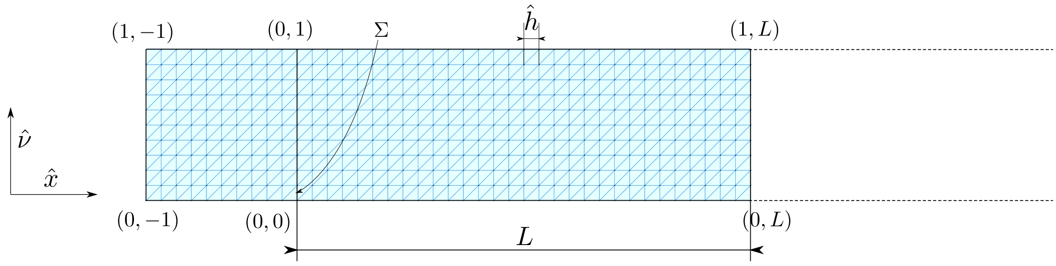


Figure 5.4: The strip truncated $\hat{Y}_L$ and the mesh T_L^h

,

- L is a large number to approximate the infinite strip $\hat{Y}_\infty$ with a truncated one defined by $\hat{Y}_L :=]0,1[\times]-1,L[$(see Figure 5.4).
- $\hat{h}$ is the maximum of size of triangles of some mesh $T_{\hat{h}}(L)$ of the truncated strip.(see Figure 5.4.) We will use this mesh in order to compute for $j \in \{1,2\}$ an approximation $\boldsymbol{M}^{\rho}_{j,\hat{h},1/L}$ of the map $\boldsymbol{M}^{\rho}_{j}$
- ΔT is a step of discretization of linear interpolation $\boldsymbol{M}^{\rho}_{j,\hat{h},1/L,\Delta T}$ of the map $\boldsymbol{M}^{\rho}_{j,\hat{h},1/L}$ for all $j \in \{1,2\}$. We have chosen to interpolate in order to reduce the number of solutions of cell's problem.

5.3.1 The discrete problem

For $i = 0,1,2$, the function $u^{\boldsymbol{h}}_{i,\delta}$ defined by the unique element of V_h such that for all $v_h \in V_h$ we have:

$$a^{\delta}_{i,\boldsymbol{h}}(u^{\boldsymbol{h}}_{i,\delta}, v_h) = (f_h, v_h)_{L^2(\Omega_h)}. \tag{5.3.1}$$

In this formulation the sesquilinear form $a^{\delta}_{i,\boldsymbol{h}}$ is given for $i \in \{1,2\}$ by:

$$a^{\boldsymbol{h}}_{1,\delta} := a^h_0 + \delta a^{\boldsymbol{h}}_1 \quad \text{and} \quad a^{\boldsymbol{h}}_{2,\delta} := a^h_0 + \delta a^{\boldsymbol{h}}_1 + \delta^2 a^{\boldsymbol{h}}_2,$$

where:

- a^h_0 is defined for $(u,v) \in H^1(\Omega_h) \times H^1(\Omega_h)$ by:

$$a^h_0(u,v) := \int_{\Omega_h} \big(\nabla u \cdot \nabla v - k^2 u \cdot \overline{v}\big) d\Omega_h,$$

- and $a^{\boldsymbol{h}}_i$ is defined for $(u,v) \in V_h \times V_h$ and $i \in \{1,2\}$ by:

$$a^{\boldsymbol{h}}_i(u,v) := \int_{\Gamma_h} \Big(\boldsymbol{\rho}^{i-1}_{eff}(\boldsymbol{h})\nabla_{\Gamma_h} u, \nabla_{\Gamma_h} v\Big) - k^2 \mu_i u \cdot \overline{v} d\Gamma_h,$$

- $\boldsymbol{\rho}^i_{eff}(\boldsymbol{h})$ is a numerical approximation of $\boldsymbol{\rho}^i_{eff}$. These approximations are given for $x_h \in \Gamma_h$ by:

$$\begin{cases} \boldsymbol{\rho}^0_{eff}(\boldsymbol{h})(x_h) := \overline{\rho}_0 - I_{\Delta T}\boldsymbol{M}^{\rho}_{0,\hat{h},1/L}\big(g_h(x_h)\big), \\ \quad \boldsymbol{\rho}^1_{eff}(x_h) := -c_h(x_h)\overline{\rho}_1 + c_h(x_h) I_{\Delta T}\boldsymbol{M}^{\rho}_{1,\hat{h},1/L}\big(g_h(x_h)\big). \end{cases}$$

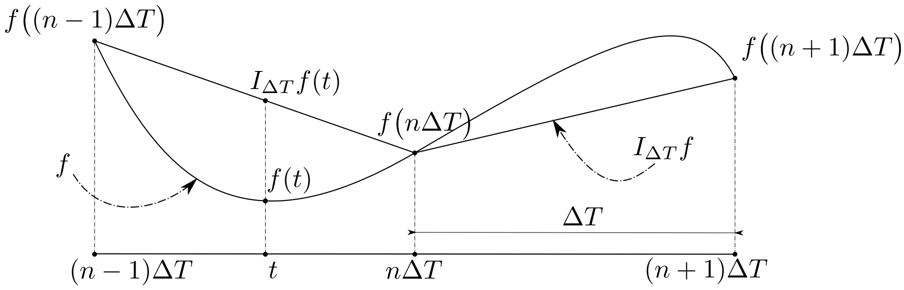


Figure 5.5: Illustration of $I_{\Delta T}$

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- $g_h, c_h : \Omega_h \mapsto \mathbb{R}_+^*$ are elements of V_h such that for all j:

$$g_h(\Gamma_j) = |V_j|^{-2} \quad \text{and} \quad c_h(\Gamma_j) := \det(A_j, V_j)/|V_j|^2$$

  where we posed:

$$V_j := \frac{P\big((j+1)\cdot h\big) - P\big((j-1)\cdot h\big)}{2\cdot h} \quad \text{and} \quad A_j := \frac{P\big((j+1)\cdot h\big) - 2\cdot P\big(j\cdot h\big) + P\big((j-1)\cdot h\big)}{h^2}.$$

- $I_{\Delta T}$ is the classical linear interpolator of step ΔT(See Figure 5.3.1).

Now let us explain how we compute the maps $\boldsymbol{M}^{\rho}_{0,\hat{h},1/L}$ and $\boldsymbol{M}^{\rho}_{1,\hat{h},1/L}$. These maps are defined for $g \in \mathbb{R}_+^*$ by:

$$\boldsymbol{M}^{\rho}_{0,\hat{h},1/L}(g) := g\cdot \int_{\hat{Y}_\infty} \hat{\rho}_{\hat{h}}(\hat{x},\hat{\nu}) \left(g\big|\partial_{\hat{x}} w_{\hat{h},1/L}\big(g;\hat{x},\hat{\nu}\big)\big|^2 + \big|\partial_{\hat{\nu}} w_{\hat{h},1/L}\big(g;\hat{x},\hat{\nu}\big)\big|^2 \right) d\hat{x}d\hat{\nu},$$

and:

$$\begin{aligned}\boldsymbol{M}^{\rho}_{1,\hat{h},1/L}\big(g\big) &:= g\cdot \int_{\hat{Y}_\infty} \hat{\rho}_{\hat{h}}(\hat{x},\hat{\nu}) \left(-g\big|\partial_{\hat{x}} w_{\hat{h},1/L}\big(g;\hat{x},\hat{\nu}\big)\big|^2 + \big|\partial_{\hat{\nu}} w_{\hat{h},1/L}\big(g;\hat{x},\hat{\nu}\big)\big|^2 \right) d\hat{x}d\hat{\nu} \\ &\quad + 2g \int_{\hat{Y}_\infty} \hat{\rho}_{\hat{h}}(\hat{x},\hat{\nu})\hat{\nu}\partial_{\hat{x}} w_{\hat{h},1/L}\big(g;\hat{x},\hat{\nu}\big) d\hat{x}d\hat{\nu}.\end{aligned}$$

Here $\hat{\rho}_{\hat{h}}$ is an approximation of $\hat{\rho}$ such that:

$$\|\hat{\rho}_{\hat{h}} - \hat{\rho}\|_{L^\infty(\hat{Y}_+)} \leq Ch,$$

where C is independent of h.

5.3.2 Numerical approximation of the map $w(g,\cdot)$ with the finite element method

To introduce the function $w_{\hat{h},1/L}\big(g;\cdot\big)$, we need to introduce the space $H^1_\#(\hat{Y}_L)$ constituted of functions of $H^1(\hat{Y}_L)$ one periodic on $\hat{x}$. Furthermore we provide this last space with the subspace of discretization $V^{L,\hat{h}}_\# \subset H^1_\#(\hat{Y}_L)$ of P_1 function on $T_{\hat{h}}(L)$ one periodic on

$\hat{x}$. Then the function $w_{\hat{h},1/L}(g;\cdot)$ is defined (up to a constant) by the unique solution of: Find $w_{\hat{h},1/L}(g;\cdot) \in V_{\#}^{L,\hat{h}}$ such that for all $v_{\hat{h}} \in V_{\#}^{L,\hat{h}}$ we have:

$$a_{\hat{h},1/L}(w_{\hat{h},1/L}(g;\cdot), v_{\hat{h}}) = l_{\hat{h},1/L}(w_{\hat{h}}). \tag{5.3.2}$$

In this variational formulation the sesquilinear form $a_{\hat{h},1/L}$ and the linear form $l_{\hat{h},1/L}$ are defined for $(v_{\hat{h}}, w_{\hat{h}}) \in V_{\#}^{L,\hat{h}} \times V_{\#}^{L,\hat{h}}$ by:

$$a_{\hat{h},1/L}(v_{\hat{h}}, w_{\hat{h}}) := \int_{\hat{Y}_\infty} \hat{\rho}_{\hat{h}}(\hat{x},\hat{\nu}) \left(g\partial_{\hat{x}} v_{\hat{h}}(\hat{x},\hat{\nu}) \partial_{\hat{x}} w_{\hat{h}}(\hat{x},\hat{\nu}) + \partial_{\hat{\nu}} v_{\hat{h}}(\hat{x},\hat{\nu}) \partial_{\hat{\nu}} w_{\hat{h}}(\hat{x},\hat{\nu}) \right) d\hat{x} d\hat{\nu},$$

and $l_{\hat{h},1/L}(w_{\hat{h}}) := \displaystyle\int_{\hat{Y}_L} \hat{\rho}_{\hat{h}}(\hat{x},\hat{\nu}) \partial_{\hat{x}} v(\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu}$.

5.4 Convergence of the method

Since Γ is a smooth curve then we have $\Omega \neq \Omega_h$. Hence the function u_i^δ and $u_{i,\delta}^{\boldsymbol{h}}$ have not the same domains definition. However from [28, Finite Element Methods for Second Order Problems Posed over Curved Domains] and [55, Approximation par éléments finis isoparamétriques dans les domaines à bords courbes] we get the following result:

Proposition 5.4.1. *If vertex of the discretization Γ_h of Γ belongs to Γ then there exists a sequence of bijective functions $\{\phi_h : \Omega \mapsto \Omega^h\}_{h>0}$ such that for all $h > 0$ we have:*

$$\phi_h(\Omega) = \Omega^h \quad \textit{and} \quad \phi_h(\Gamma) = \Gamma^h,$$

and there exists $C > 0$ such that for all $h > 0$ we have the following estimations:

$$\|\mathbb{I} - \phi_h\|_{W^{0,\infty}(\Omega)} \leq C \cdot h^2 \quad \textit{and} \quad \|\mathbb{I} - \phi_h\|_{W^{1,\infty}(\Gamma)} \leq C \cdot h.$$

Moreover this last sequence of functions satisfies for all $h > 0$:

$$\sup(\mathbb{I} - \phi_h) \subset \Big\{ x \in \Omega_h,\ \operatorname{dist}(x,\Gamma) \leq h \Big\}.$$

We refer the reader to Figure 5.3 for an illustration of this result. Thus we now can state the following result that we now prove:

Theorem 5.4.2. *There exists $C > 0$ independent of $\boldsymbol{h}$ such that for all $i = 0, 1, 2$ the following estimate holds:*

$$\delta^{\frac{1}{2}} \|u_i^\delta - u_{i,\boldsymbol{h}}^\delta \circ \phi_h\|_{H^1(\Gamma)} + \|u_i^\delta - u_{i,\boldsymbol{h}}^\delta \circ \phi_h\|_{H^1(\Omega)} \leq C\Big(h + \delta^{\frac{1}{2}} \epsilon(\boldsymbol{h})\Big).$$

This result is a direct consequence of Proposition 5.4.1 and upcomes results: Lemma 5.4.3, Lemma 5.4.4, Lemma 5.4.5, Lemma 5.4.7 and Lemma 5.4.8. We prove now these results.

5.4.1 Stability of the discretization of the effective boundary conditions

We introduce for convenience the norm N_h^δ defined for $u \in H^1(\Gamma_h) \cap H^1(\Omega_h)$ by:

$$N_h^\delta(u) := \delta^{\frac{1}{2}} \|u\|_{H^1(\Gamma_h)} + \|u\|_{H^1(\Omega_h)}.$$

We have a useful result of stability of our numerical approximation:
Lemma 5.4.3. *There exists $h_0 > 0$ and $\eta_0 > 0$ independent of δ and $\boldsymbol{h}$ such that for all $x_h \in V_h$ we have:*

$$|\boldsymbol{h}| < h_0 \quad \Rightarrow \quad N_h^\delta(x_\eta) \le \eta_0 \sup_{\substack{y_h \in V_h \\ N_h^\delta(y_h)=1}} a_{i,\boldsymbol{h}}^\delta(x_h, y_h).$$

Proof. We prove this result by contradiction. The contradiction of our stated result implies that for all $\eta > 0$ there exists $\delta_\eta > 0$, $\boldsymbol{h}'$ with $|\boldsymbol{h}'| < \eta$ and $x_\eta \in V_{\boldsymbol{h}_0'}$ such that:

$$N_{\boldsymbol{h}_0'}^{\delta_\eta}(x_\eta) = 1 \quad \text{and} \quad \sup_{\substack{y_\eta \in V_{\boldsymbol{h}_0'} \\ N_{\boldsymbol{h}_0'}^{\delta_\eta}(y_\eta)=1}} \left| a_{i,\boldsymbol{h}'}^{\delta_\eta}(x_\eta, y_\eta) \right| \le \eta, \tag{5.4.3}$$

and let us prove that this last proposition is absurd.

First let us prove that the set of the weak limit point in the space $H^1(\Omega)$ of the sequence $(x_\eta \circ \phi_{\boldsymbol{h}_0'})_{\eta>0}$ is reduced to the singleton $\{0\}$. Let x_0 be a weak limit point of this last sequence and δ_0 be a limit points of the sequence $(\delta_\eta)_{\eta>0}$.

- If $\delta_0 > 0$ then the sequence $(x_\eta \circ \phi_{\boldsymbol{h}_0'})_{\eta>0}$ is bounded in the space $V_{\mathcal{Z}}$. Therefore this last sequence weakly converge to x_0 in the space $V_{\mathcal{Z}}$ and $x_0 \in V_{\mathcal{Z}}$. Therefore combining this last convergence property with and Proposition 5.4.1 and Lemma 5.4.8 yields the following implication:

$$\delta_0 > 0 \Rightarrow \forall y \in C^\infty(\overline{\Omega}),\ \lim_{h \to 0} a_{i,\boldsymbol{h}'}^{\delta_\eta}(x_\eta, y \circ \phi_{\boldsymbol{h}_0'}^{-1}) = a_i^{\delta_0}(x_0, y). \tag{5.4.4}$$

- If $\delta_0 = 0$ then the quantity $(x_\eta \circ \phi_{\boldsymbol{h}_0'})_{\eta>0}$ could eventually not be bounded in the space $H^1(\Gamma)$. However from $N_{\boldsymbol{h}_0'}^{\delta_\eta}(x_\eta) = 1$ we get that $\delta_\eta^{\frac{1}{2}} x_\eta \circ \phi_{\boldsymbol{h}_0'}$ is bounded in the space $H^1(\Gamma)$ which leads that for all $j \in \{1, 2\}$ the quantity $\delta_\eta^{\frac{1}{2}} a_j^{\boldsymbol{h}}(x_\eta, y \circ \phi_{\boldsymbol{h}_0'}^{-1})$ is bounded which leads to

$$\lim_{\eta \to 0} \sum_{j=1}^{i} \delta_\eta^j a_j^{\boldsymbol{h}'}(x_\eta, y \circ \phi_{\boldsymbol{h}_0'}^{-1}) = 0.$$

Combining this last convergence with the weak convergence of $(x_\eta)_{\boldsymbol{h}}$ in the space in $H^1(\Omega)$ and Proposition 5.4.1 yields the following implication:

$$\delta_0 = 0 \Rightarrow \forall y \in C^\infty(\overline{\Omega}),\ \lim_{h \to 0} a_{i,\boldsymbol{h}}^{\delta_\eta}(x_\eta, y) = a_i^0(x_0, y). \tag{5.4.5}$$

Now let us prove that we have for all smooth function y:

$$\lim_{\eta\to 0} a_i^{\delta_\eta}(x_\eta, y\circ\phi_{\boldsymbol{h}'_0}^{-1}) = 0. \tag{5.4.6}$$

Indeed since y is a smooth function then we have existence of $y_\eta \in V_{\boldsymbol{h}'_0}$ such that:

$$N_{\boldsymbol{h}'_0}^{\delta_\gamma}(y\circ\phi_{\boldsymbol{h}'_0}^{-1} - y_\eta) \leq C\eta,$$

where $C > 0$ is independent of η which leads to $\lim\limits_{\eta\to 0} a_i^{\delta_\eta}(x_\gamma, y\circ\phi_{\boldsymbol{h}'_0}^{-1} - y_\eta) = 0$. Therefore to prove (5.4.6) it remains to show that:

$$\lim_{\eta\to 0} a_i^{\delta_h}(x_\eta, y_\eta) = 0,$$

which is a direct consequence of (5.4.3). Thus combining (5.4.4), (5.4.5) with (5.4.6) leads to that for all smooth function y we have:

$$a_i^{\delta_0}(x_0, y) = 0.$$

Therefore x_0 is a solution of the approximate problem order of i with $\delta = \delta_0$ with $f = 0$ and using that this last problem is well posed implies that $x_0 = 0$.

Now we succeed to prove that x_η weakly converges to 0 in the space $H^1(\Omega)$. Therefore we get that $(x_\eta \circ \phi_{\boldsymbol{h}'_0})_{\eta>0}$ strongly converges to 0 in the space $L^2(\Omega)$. Moreover we have it is clear that there exists $C > 0$ independent of η we have:

$$N_{\boldsymbol{h}'_0}^{\delta}(x_\eta) \leq a_i^{\delta_h}(x_\eta, x_\eta) + C\|x_\eta\|_{L^2(\Omega)}^2$$

Combining this last estimate with (5.4.3) yields which bring a contradiction contradiction with the assumption $N_{\boldsymbol{h}'_0}^{\delta}(x_\eta) = 1$ and therefore ends our proof. □

5.4.2 Decomposition of the error

We introduce the two following quantities:

- The interpolation error is defined by:

$$\mathcal{D}_{\text{interp}} := N_h^\delta\left(u_i^\delta\circ\phi_h^{-1} - \Pi_{T_h} u_i^\delta\circ\phi_h^{-1}\right).$$

  This error measures how $u_i^\delta\circ\phi_h^{-1}$ fails to be in the space V_h.

- The consistency error is defined by:

$$\mathcal{D}_{\text{consistence}} := \sup_{\substack{y_h\in V_h\\ N_h^\delta(y_h)=1}} a_{i,\boldsymbol{h}}^\delta\left(u_i^\delta\circ\phi_h^{-1}, y_h\right) - a_i^\delta\left(u_i^\delta, y_h\circ\phi_h\right).$$

  This error is due to error of approximation of the map w and our geometry.

Lemma 5.4.4. *One has*

$$N_h^\delta\left(u_i^\delta\circ\phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta\right) \leq \mathcal{D}_{\text{interp}} + \mathcal{D}_{\text{consistence}}.$$

Proof. Since $\Pi_{T_h} u_i^\delta \circ \phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta \in V_h$, we have thanks to Lemma 5.4.3 that:

$$\begin{aligned}
N_h^\delta\left(u_i^\delta \circ \phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta\right) &\leq N_h^\delta\left(\Pi_{T_h} u_i^\delta \circ \phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta\right) + N_h^\delta\left(u_i^\delta \circ \phi_h^{-1} - \Pi_{T_h} u_i^\delta \circ \phi_h^{-1}\right), \\
&\leq \eta_0 \sup_{\substack{y_h \in V_h \\ N_h^\delta(y_h)=1}} a_{i,\boldsymbol{h}}^\delta\left(\Pi_{T_h} u_i^\delta \circ \phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta, y_h\right) + \mathcal{D}_{\text{interp}}, \\
&\leq \eta_0 \sup_{\substack{y_h \in V_h \\ N_h^\delta(y_h)=1}} a_{i,\boldsymbol{h}}^\delta\left(u_i^\delta \circ \phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta, y_h\right) + \mathcal{D}_{\text{interp}}.
\end{aligned}$$

Combining this last estimate with (5.3.1) yields:

$$\begin{aligned}
N_h^\delta\left(u_i^\delta \circ \phi_h^{-1} - u_{i,\boldsymbol{h}}^\delta\right) &\leq \eta_0 \sup_{\substack{y_h \in V_h \\ N_h^\delta(y_h)=1}} a_{i,\boldsymbol{h}}^\delta\left(u_i^\delta \circ \phi_h^{-1}, y_h\right) - (f, y_h \circ \phi_h) + \mathcal{D}_{\text{interp}}, \\
&\leq \eta_0 \sup_{\substack{y_h \in V_h \\ N_h^\delta(y_h)=1}} a_{i,\boldsymbol{h}}^\delta\left(u_i^\delta \circ \phi_h^{-1}, y_h\right) - a_i^\delta\left(u_i^\delta, y_h \circ \phi_h\right) + \mathcal{D}_{\text{interp}},
\end{aligned}$$

which concludes the proof. □

5.4.3 Estimate of the interpolation error

Lemma 5.4.5. *There exists $C > 0$ independent of $\boldsymbol{h}, \delta$ such that:*

$$\mathcal{D}_{\text{interp}} \leq C, \tag{5.4.7}$$

This result is a direct consequence of:

Proposition 5.4.6. *There exists $C > 0$ such that for all $n \leq m_\Gamma$ and $\delta > 0$ we have for all $i \in \{1, 2\}$:*

$$\delta^{\frac{1}{2}} \|u_i^\delta\|_{H^{n+1}(\Gamma)} + \|u_i^\delta\|_{H^{n+1}(\Omega)} \leq C. \tag{5.4.8}$$

Proof. To simplify the writing of the proof we assume that $k = 0$ because the generalization for $k \neq 0$ is trivial. By using chart and unit partition of unity we can assume that Γ is $\mathbb{R} \times \{0\}$, $\Omega = \mathbb{R} \times]0, 1[$ and u_i^δ is the unique solution of: Find $u_i^\delta \in V_{\mathcal{Z}}$ such that:

$$\operatorname{div}\left(P \nabla u_i^\delta Q u^\delta\right) = f \text{ in } \Omega \quad \text{and} \quad \partial_\nu u_i^\delta = \delta \operatorname{div}_\Gamma\left(\rho_i^\delta \nabla_\Gamma u_i^\delta\right) \text{ on } \Gamma,$$

where P is a C^{m_Γ} matrix valued function of the form $P = \operatorname{diag}(1, P_{xx})$ and $\{P_j\}_j \in C^{m_\Gamma}(\Gamma)$ such that there exist $P_j^c > 0$ such that for all $x \in \Gamma$ we have $P_j(x) \geq P_j^c$. We will prove the estimate (5.4.8) by a recurrence on n. Thanks to stability result Lemma 4.5.4, the result the result is trivial for $n = 0$. Let n such that (5.4.8) is true and let us prove the following estimate:

$$\delta^{\frac{1}{2}} \|u_i^\delta\|_{H^{n+2}(\Gamma)} + \|u^\delta\|_{H^{n+2}(\Omega)} \leq C. \tag{5.4.9}$$

First let us prove that we have $\partial_x^{\alpha+1} u^\delta \in H^1(\Omega) \cap H^1(\Gamma)$ and the following estimate:

$$\delta^{\frac{1}{2}} \|\partial_x u_i^\delta\|_{H^{n+1}(\Gamma)} + \|\partial_x u^\delta\|_{H^{n+1}(\Omega)} \leq C. \tag{5.4.10}$$

Thanks to the Leibniz formula we get that for all $\alpha \leq n$ that $\partial_x^\alpha u^\delta$ is the unique solution of: Find $\partial_x^\alpha u^\delta \in H^1(\Omega) \cap H^1(\Gamma)$ such that we have:

$$\operatorname{div}\left(P \nabla \partial_x^\alpha u^\delta\right) = f_\alpha^\delta \quad \text{and} \quad \partial_\nu \partial_x^\alpha u^\delta - \delta \operatorname{div}_\Gamma\left(\rho_i^\delta \nabla_\Gamma \partial_x^\alpha u^\delta\right) = g_\alpha^\delta, \tag{5.4.11}$$

where we defined the following quantity:

$$f_\alpha^\delta := \partial_x^\alpha f - \sum_{\alpha'<\alpha} \operatorname{div}\left((\partial_x^{\alpha-\alpha'} P)\nabla \partial_x^{\alpha'} u^\delta\right) \quad \text{and} \quad g_\alpha^\delta := \sum_{\alpha'<\alpha} \delta \operatorname{div}_\Gamma\left(\left(\partial_x^{\alpha-\alpha'} \rho_i^\delta\right) \nabla_\Gamma \partial_x^{\alpha'} u^\delta\right).$$

Thanks to the recurrence hypothesis (5.4.8) we get that $f_\alpha^\delta \in L^2(\Omega)$ and $g_\alpha^\delta \in L^2(\Gamma)$ with the following estimate:

$$\left\|f_\alpha^\delta\right\|_{L^2(\Omega)} \le C \quad \text{and} \quad \left\|g_\alpha^\delta\right\|_{L^2(\Gamma)} \le C\delta^{\frac{1}{2}} \tag{5.4.12}$$

Now we introduce for $h > 0$ the translation operator T_h defined for all function u defined on Ω and $(x, y) \in \Omega$ by $T_h u(x, y) = u(x + h, y)$ in order to write for all $h > 0$

$$\begin{aligned} T_h f_\alpha^\delta - f_\alpha^\delta &= \operatorname{div}\left((T_h P)\nabla \partial_x^\alpha T_h u^\delta\right) - \operatorname{div}\left(P \nabla \partial_x^\alpha u^\delta\right), \\ &= \operatorname{div}\left((T_h P - P)\nabla \partial_x^\alpha T_h u^\delta\right) + \operatorname{div}\left(P\nabla (T_h \partial_x^\alpha u^\delta - \partial_x^\alpha u^\delta)\right), \end{aligned}$$

which leads to:

$$\forall h > 0,\ \operatorname{div}\left(P \nabla D_h \partial_x^\alpha u^\delta\right) = D_h f_\alpha^\delta - \operatorname{div}\left((D_h P)\nabla \partial_x^\alpha u^\delta\right), \tag{5.4.13}$$

where we defined the operator $D_h := (T_h - \mathbb{I})/h$. Moreover we have with the same idea that:

$$\forall h > 0,\ \partial_\nu \partial_x^\alpha D_h u^\delta - \delta \operatorname{div}_\Gamma\left(\rho_i^\delta \nabla_\Gamma \partial_x^\alpha D_h u^\delta\right) = g_\alpha^\delta + \delta \operatorname{div}_\Gamma\left((D_h \rho_i^\delta) \nabla_\Gamma \partial_x^\alpha u^\delta\right). \tag{5.4.14}$$

Now let us prove that we have existence of $C > 0$ independent of $h > 0$ and $\delta > 0$ such that the following estimate holds:

$$\left\|D_h f_\alpha^\delta - \operatorname{div}\left((D_h P)\nabla \partial_x^\alpha u^\delta\right)\right\|_{(H^1(\Omega))^\dagger} + \delta^{-\frac{1}{2}} \left\|D_h g_\alpha^\delta + \delta \operatorname{div}_\Gamma\left((D_h \rho_i^\delta) \nabla_\Gamma \partial_x^\alpha u^\delta\right)\right\|_{H^{-1}(\Gamma)} \le C. \tag{5.4.15}$$

Indeed thanks to the recurrence hypothesis (5.4.8) we get that:

$$\left\|\partial_x^\alpha u^\delta\right\|_{H^1(\Omega)} \le C \quad \text{and} \quad \left\|\partial_x^\alpha u^\delta\right\|_{H^1(\Gamma)} \le C\delta^{-\frac{1}{2}}. \tag{5.4.16}$$

Moreover the regularity on P and ρ_i^δ imply existence of $C > 0$ independent of $\delta > 0$ such that:

$$\|\partial_x P\|_{L^\infty(\Omega)} \le C \quad \text{and} \quad \|\partial_x \rho_i^\delta\|_{L^\infty(\Gamma)} \le C,$$

which leads combined with (5.4.16) to:

$$\left\|\operatorname{div}\left((\partial_x P)\nabla \partial_x^\alpha u^\delta\right)\right\|_{(H^1(\Omega))^\dagger} \le C \quad \text{and} \quad \left\|\operatorname{div}_\Gamma\left((\partial_x \rho_i^\delta) \nabla_\Gamma \partial_x^\alpha u^\delta\right)\right\|_{H^{-1}(\Gamma)} \le C\delta^{-\frac{1}{2}}.$$

Therefore combining these last estimate with (5.4.12) yields:

$$\underbrace{\left\|\partial_x f_\alpha^\delta - \operatorname{div}\left((\partial_x P)\nabla \partial_x^\alpha u^\delta\right)\right\|_{(H^1(\Omega))^\dagger}}_{A} + \delta^{-\frac{1}{2}} \underbrace{\left\|\partial_x g_\alpha^\delta + \delta \operatorname{div}_\Gamma\left((\partial_x \rho_i^\delta) \nabla_\Gamma \partial_x^\alpha u^\delta\right)\right\|_{H^{-1}(\Gamma)}}_{B} \le C,$$

and combining this last estimate with the mean value theorem end the proof of (5.4.15).

Moreover combing (5.4.14) and (5.4.13) leads to

$$\left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Omega)}^2+\delta\left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Gamma)}^2 \le A\left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Omega)}+\delta B\left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Gamma)}^2,$$
$$\le\sqrt{A^2+\delta B^2}\sqrt{\|D_h\partial_x^\alpha u^\delta\|_{H^1(\Omega)}+\delta\left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Gamma)}^2},$$

and then combining this last estimate with (5.4.15) leads to the existence of $C>0$ independent of δ such that the following estimate holds:

$$\forall h>0,\ \left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Omega)}+\delta^{\frac{1}{2}}\left\|D_h\partial_x^\alpha u^\delta\right\|_{H^1(\Gamma)}\le C.$$

Therefore there exists an element $Q^\delta\in H^1(\Omega)\cap H^1(\Gamma)$ such that there exists $C>0$ independent of δ such that

$$\left\|Q^\delta\right\|_{H^1(\Omega)}+\delta^{\frac{1}{2}}\left\|Q^\delta\right\|_{H^1(\Gamma)}\le C, \tag{5.4.17}$$

and such that we have the weak convergence in the sense that for all smooth function ϕ :

$$\lim_{h\to 0}\left(D_h\partial_x^\alpha u^\delta,\phi\right)_{L^2(\Omega)}=\left(Q^\delta,\phi\right)_{L^2(\Omega)}\quad\text{and}\quad\lim_{h\to 0}\left(D_h\partial_x^\alpha u^\delta,\phi\right)_{L^2(\Gamma)}=\left(Q^\delta,\phi\right)_{L^2(\Gamma)}. \tag{5.4.18}$$

Now let us prove that in the sense of the distribution

$$\partial_x^{\alpha+1}u^\delta=Q^\delta \text{ in } D'(\Omega)\quad\text{and}\quad\partial_x^{\alpha+1}u^\delta=Q^\delta\text{ in } D'(\Gamma). \tag{5.4.19}$$

Indeed for all smooth function $\phi\in C^\infty(\overline{\Omega})$ we have the following strong convergence in the space $C^\infty(\overline{\Omega})$

$$\lim_{h\to 0}D_h\phi=\partial_x\phi.$$

Therefore combining with (5.4.18) yields that for all $\phi\in D(\Omega)$ we have

$$-\left(\partial_x^\alpha u^\delta,\partial_x\phi\right)_{L^2(\Omega)}=\left(Q^\delta,\phi\right)_{L^2(\Omega)}\quad\text{and}\quad-\left(\partial_x^\alpha u^\delta,\partial_x\phi\right)_{L^2(\Gamma)}=\left(Q^\delta,\phi\right)_{L^2(\Gamma)}.$$

which end the proof of (5.4.19). Thus combining (5.4.17) with (5.4.19) end the proof of (5.4.10).

Since we success to prove (5.4.10) we first get the existence of $C>0$ independent of δ such that following estimates hold:

$$\delta^{\frac{1}{2}}\|u_i^\delta\|_{H^{n+2}(\Gamma)}\le C\quad\text{and}\quad\forall q\le n+1,\ \|\partial_x^{n+2-q}\partial_\nu^q u^\delta\|_{L^2(\Omega)}\le C. \tag{5.4.20}$$

Thus in order to end whole the proof it is sufficient to prove that $\partial_\nu^{n+2}u^\delta\in L^2(\Omega)$ and existence of $C>0$ independent of δ such that:

$$\left\|\partial_\nu^{n+2}u^\delta\right\|_{L^2(\Omega)}\le C. \tag{5.4.21}$$

This result is a direct consequence of taking $\alpha=(n,0)$ in (5.4.11) leads to

$$\partial_\nu^{n+2}u^\delta+\partial_x\left(P_{xx}(\partial_x\partial_\nu^n u^\delta)\right)=f_{(n,0)}^\delta$$

which leads to:

$$\partial_\nu^{n+2}u^\delta=\underbrace{f_{(n,0)}^\delta}_{\text{in } L^2(\Omega)\text{ thanks to (5.4.12)}}-\underbrace{\partial_x P_{xx}\partial_x\partial_\nu^n u^\delta}_{\text{in } L^2(\Omega)\text{ thanks to (5.4.20)}}.$$

5.4.4 Decomposition of the consistency error

Lemma 5.4.7. *One has the following decomposition of the error estimate:*

$$\mathcal{D}_{\text{consistence}} \leq C\left(\left\|\mathrm{D}\,\phi_h^{-1}\,\mathrm{D}\,\phi_h^{-\dagger}\big|\det(\mathrm{D}\,\phi_h)\big| - \mathbb{I}\right\|_{L^\infty(\Omega)} + \max_{j\in\{0,1\}}\left\|\boldsymbol{\rho}^j_{eff}\circ\phi_h^{-1} - \boldsymbol{\rho}^j_{eff}(\boldsymbol{h})\right\|_{L^\infty(\Gamma_h)}\right)$$

Proof. To simplify the writing of the proof we assume that $k=0$ because the generalization for $k\neq 0$ is obvious. Thanks to variable change formula for the gradient operator we get that for all $y_h\in V_h$ we have:

$$\begin{aligned}
a^\delta_{i,\boldsymbol{h}}\left(u^\delta_i\circ\phi_h^{-1}, y_h\right) =& \sum_{j=1}^{i}\delta^j\int_{\Gamma_h}\left(\boldsymbol{\rho}^j_{eff}(\boldsymbol{h})\nabla_{\Gamma_h}(u^\delta_i\circ\phi_h^{-1}), \nabla_{\Gamma_h}y_h\right)d\Gamma_h + \int_{\Omega_h}\Big(\nabla(u^\delta_i\circ\phi_h^{-1}), \nabla y_h\Big)d\Omega_h,\\
=& \sum_{j=1}^{i}\delta^j\int_{\Gamma}\Big(\boldsymbol{\rho}^j_{eff}(\boldsymbol{h})\circ\phi_h\cdot\mathrm{D}\,\phi_h^{-\dagger}\nabla_\Gamma u^\delta_i, \mathrm{D}\,\phi_h^{-\dagger}\nabla_\Gamma(y_h\circ\phi_h)\Big)\,\big|\det(\mathrm{D}\,\phi_h)\big|d\Gamma +\\
&\int_\Omega\Big(\mathrm{D}\,\phi_h^{-\dagger}\nabla u^\delta_i, \mathrm{D}\,\phi_h^{-\dagger}\nabla(y_h\circ\phi_h)\Big)\big|\det(\mathrm{D}\,\phi_h)\big|d\Omega,\\
=& \sum_{j=1}^{i}\delta^j\int_{\Gamma}\Big(\mathrm{D}\,\phi_h^{-1}\cdot\boldsymbol{\rho}^j_{eff}(\boldsymbol{h})\circ\phi_h\cdot\mathrm{D}\,\phi_h^{-\dagger}\nabla_\Gamma u^\delta_i, \nabla_\Gamma(y_h\circ\phi_h)\Big)\,\big|\det(\mathrm{D}\,\phi_h)\big|d\Gamma +\\
&\int_\Omega\Big(\mathrm{D}\,\phi_h^{-1}\,\mathrm{D}\,\phi_h^{-\dagger}\nabla u^\delta_i, \nabla(y_h\circ\phi_h)\Big)\big|\det(\mathrm{D}\,\phi_h)\big|d\Omega.
\end{aligned}$$

Therefore for all $y_h\in V_h$ we have:

$$a^\delta_{i,\boldsymbol{h}}\left(u^\delta_i\circ\phi_h^{-1}, y_h\right) - a^\delta_i\left(u^\delta_i, y_h\circ\phi_h\right) = \int_\Gamma\left(D_\Gamma\nabla_\Gamma u^\delta_i, \nabla_\Gamma(y_h\circ\phi_h)\right)d\Gamma + \int_\Omega\Big(D_\Omega\nabla u^\delta_i, \nabla(y_h\circ\phi_h)\Big),$$

where we defined on Γ and Ω the following tensors field

$$\begin{cases}
D_\Gamma := \displaystyle\sum_{j=1}^{i}\delta^j\left(\mathrm{D}\,\phi_h^{-1}\cdot\boldsymbol{\rho}^j_{eff}(\boldsymbol{h})\circ\phi_h\cdot\mathrm{D}\,\phi_h^{-\dagger}\big|\det(\mathrm{D}\,\phi_h)\big| - \boldsymbol{\rho}^j_{eff}\right),\\
D_\Omega := \mathrm{D}\,\phi_h^{-1}\,\mathrm{D}\,\phi_h^{-\dagger}\big|\det(\mathrm{D}\,\phi_h)\big| - \mathbb{I}
\end{cases}$$

Therefore we have for all $y_h\in V_h$

$$\begin{aligned}
a^\delta_{i,\boldsymbol{h}}\left(u^\delta_i\circ\phi_h^{-1}, y_h\right) - a^\delta_i\left(u^\delta_i, y_h\circ\phi_h\right) \leq& \,\delta\left\|D_\Gamma\right\|_{L^\infty(\Gamma)}\left\|u^\delta_i\right\|_{H^1(\Gamma)}\left\|y_h\right\|_{H^1(\Gamma)} +\\
&\left\|D_\Omega\right\|_{L^\infty(\Omega)}\left\|u^\delta_i\right\|_{H^1(\Omega)}\left\|y_h\right\|_{H^1(\Omega)},\\
\leq& \left(\delta^{\frac{1}{2}}\left\|D_\Gamma\right\|_{L^\infty(\Gamma)}\left\|u^\delta_i\right\|_{H^1(\Gamma)} + \left\|D_\Omega\right\|_{L^\infty(\Omega)}\left\|u^\delta_i\right\|_{H^1(\Omega)}\right)N^\delta_h(y_h).
\end{aligned}$$

Thus applying Proposition 5.4.6 with $n=1$ and the trace theorem we get that the quantity $\delta^{\frac{1}{2}}\|u^\delta_i\|_{H^1(\Gamma)}$ is bounded when $\delta\to 0$. Therefore we have:

$$a^\delta_{i,\boldsymbol{h}}\left(u^\delta_i\circ\phi_h^{-1}, y_h\right) - a^\delta_i\left(u^\delta_i, y_h\circ\phi_h\right) \leq C\left(\delta^{\frac{1}{2}}\left\|D_\Gamma\right\|_{L^\infty(\Gamma)} + \left\|D_\Omega\right\|_{L^\infty(\Omega)}\right)N^\delta_h(y_h), \quad (5.4.22)$$

where $C > 0$ is a constant independent of h and δ. Moreover, we can prove that:

$$\left(\delta^{\frac{1}{2}} \|D_\Gamma\|_{L^\infty(\Gamma)} + \|D_\Omega\|_{L^\infty(\Omega)}\right) N_h^\delta(y_h) \leq C \left(\left\| \mathrm{D}\,\phi_h^{-1}\,\mathrm{D}\,\phi_h^{-\dagger} \big| \det(\mathrm{D}\,\phi_h) \big| - \mathbb{I} \right\|_{L^\infty(\Omega)} \right)$$
$$+ C \left(\max_{j \in \{0,1\}} \left\| \boldsymbol{\rho}^j_{eff} \circ \phi_h^{-1} - \boldsymbol{\rho}^j_{eff}(\boldsymbol{h}) \right\|_{L^\infty(\Gamma_h)} \right).$$

Combining this with (5.4.22) concludes the proof. □

5.4.5 Error estimate of the approximation of the effective coefficient

Lemma 5.4.8. *There exists $C > 0$ independent of $h, \hat{h}, \Delta T$ and L such that the following holds:*

$$\left\| \boldsymbol{\rho}^0_{eff} \circ \phi_h^{-1} - \boldsymbol{\rho}^0_{eff}(\boldsymbol{h}) \right\|_{L^\infty(\Gamma_h)} \leq C\epsilon(\boldsymbol{h}) \quad \text{and} \quad \left\| \boldsymbol{\rho}^1_{eff} \circ \phi_h^{-1} - \boldsymbol{\rho}^1_{eff}(\boldsymbol{h}) \right\|_{L^\infty(\Gamma_h)} \leq C\epsilon(\boldsymbol{h}),$$

where we defined $\epsilon(\boldsymbol{h}) := h^2 + \hat{h} + \Delta T^2 + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}})$.

This result is a direct consequence of Proposition 5.4.12 and Proposition 5.4.13. We introduce the following interval:

$$I := [\inf(g), \sup(g)],$$

and we emphasize that $\inf(g) > 0$.

Proposition 5.4.9. *We have $w \in C^{m_\Gamma}\big(I; H^1(\hat{Y}_\infty)\big)$ and $w_{h,\hat{h},L} \in C^{m_\Gamma}\big(I; H^1(\hat{Y}_L)\big)$. Moreover there exists $C > 0$ independent of $h, \hat{h}$ and L such that the following estimate holds:*

$$\left\| w - \Pi_{T_{\hat{h}}} w \right\|_{C^{m_\Gamma}\left(I; H^1(\hat{Y}_L)\right)} \leq C\hat{h},$$

where $T_{\hat{h}}$ is the P1 interpolator on the mesh $T_{\hat{h}}(\hat{Y}_L)$.

Proof. First prove that:

$$w \in C^{m_\Gamma}\big(I; H^1(\hat{Y}_\infty)\big) \quad \text{and} \quad w_{\hat{h},1/L} \in C^{m_\Gamma}\big(I; H^1(\hat{Y}_L)\big), \tag{5.4.23}$$

in order to give a sense of the estimate appearing in our result. We introduce for convenience the operators:

$$A_{\hat{h},L} \in C^{m_\Gamma}\Big(I; \mathcal{L}\big(H^1_\#(\hat{Y}_L), H^1_\#(\hat{Y}_L)^\dagger\big)\Big) \quad \text{and} \quad A \in C^{m_\Gamma}\Big(I; \mathcal{L}\big(\mathbb{H}\big(\hat{Y}_\infty\big), \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\big)\Big)$$

defined for $t \in I$ and $(u, v, u_L, v_L) \in H^1_\#(\hat{Y}_L) \times H^1_\#(\hat{Y}_L) \times \mathbb{H}\big(\hat{Y}_\infty\big) \times \mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\begin{cases} \Big\langle A_{\hat{h},L}(t) u_L, v_L \Big\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} := \displaystyle\int_{\hat{Y}_L} \hat{\rho}_{\hat{h}} \Big(M(t)\,\widehat{\nabla}\, u_L, \widehat{\nabla}\, v_L \Big) d\hat{x} d\hat{\nu} + \int_\Sigma u_L d\hat{x} \cdot \int_\Sigma v_L d\hat{x}, \\ \langle A(t) u, v \rangle_{\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger - \mathbb{H}\left(\hat{Y}_\infty\right)} := \displaystyle\int_{\hat{Y}_\infty} \hat{\rho} \Big(M(t)\,\widehat{\nabla}\, u, \widehat{\nabla}\, v \Big) d\hat{x} d\hat{\nu} + \int_\Sigma u d\hat{x} \cdot \int_\Sigma v d\hat{x}. \end{cases}$$

Here we defined:

$$M(t) := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

The linear forms $L_{\hat{h},L} \in C^{m_\Gamma}\Big(I; H^1_\#(\hat{Y}_L)^\dagger\Big)$ and $L \in C^{m_\Gamma}\Big(I; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big)$ defined for $t \in I$ and $(v_L, v) \in H^1_\#(\hat{Y}_L) \times \mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\begin{cases} \Big\langle L_{\hat{h},L}(t), v_L \Big\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} := \displaystyle\int_{\hat{Y}_\infty} \hat{\rho}_{\hat{h}}(t;\cdot)\partial_{\hat{x}} v_L d\hat{x} d\hat{\nu}, \\ \langle L, v\rangle_{\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger - \mathbb{H}\big(\hat{Y}_\infty\big)} := \displaystyle\int_{\hat{Y}_\infty} \hat{\rho}(t;\cdot)\partial_{\hat{x}} v d\hat{x} d\hat{\nu}. \end{cases}$$

With these last definitions we get that for all t the functions $w_{\hat{h},1/L}(t;\cdot)$ and $w(t;\cdot)$ are the unique element of $V^{L,\hat{h}}_\#$ and $\mathbb{H}\big(\hat{Y}_\infty\big)$ such that: for all $(v, v_h) \in V^{L,\hat{h}}_\# \times \mathbb{H}\big(\hat{Y}_\infty\big)$ we have:

$$\begin{cases} \Big\langle A_{\hat{h},L}(t) \cdot w_{\hat{h},1/L}(t;\cdot), v_L \Big\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} = \Big\langle L_{\hat{h},L}(t), v_L \Big\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)}, \\ \langle A(t) \cdot w(t;\cdot), v\rangle_{\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger - \mathbb{H}\big(\hat{Y}_\infty\big)} = \langle L(t), v\rangle_{\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger - \mathbb{H}\big(\hat{Y}_\infty\big)}. \end{cases} \tag{5.4.24}$$

These last equations can be rewritten as below:

$$w_{\hat{h},1/L}(t;\cdot) = \Big(A^{\text{restriction}}_{h,\hat{h},L}(t)\Big)^{-1} \cdot L_{\hat{h},L}(t) \quad \text{and} \quad w(t;\cdot) = A(t)^{-1} L(t), \tag{5.4.25}$$

where $A^{\text{restriction}}_{h,\hat{h},L}(t) \in \mathcal{L}(V^{L,\hat{h}}_\#, (V^{L,\hat{h}}_\#)^\dagger)$ is the operator defined for $u, v \in V^{L,\hat{h}}_\#$ by:

$$\Big\langle A^{\text{restiction}}_{h,\hat{h},L}(t) \cdot u, v \Big\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} = \Big\langle A_{\hat{h},L}(t) \cdot u, v \Big\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)}.$$

Moreover thanks to the assumption (4.1.12) the operators $A^{\text{restiction}}_{h,\hat{h},L}$ and A are uniformly coercive on I which implies that their inverse belong to $C^{m_\Gamma}\Big(\mathcal{L}(V^{L,\hat{h}}_\#)^\dagger, V^{L,\hat{h}}_\#\Big)$ and $C^{m_\Gamma}\Big(\mathbb{H}\big(\hat{Y}_\infty\big)^\dagger, \mathbb{H}\big(\hat{Y}_\infty\big)\Big)$. Thus combing this last property with (5.4.25) and the regularity of L and $L_{\hat{h},L}$ end the proof of (5.4.23). Thus now we prove that

$$\Big\| w - \Pi_{T_{\hat{h}}} w \Big\|_{C^{m_\Gamma}\big(I; H^1(\hat{Y}_L)\big)} \leq C\hat{h}.$$

Thanks to [28, Theorem 15.3] a sufficient condition for this last estimate is to prove that:

$$w \in C^{m_\Gamma}\big(I; H^2(\hat{Y}_0)\big) \cap C^{m_\Gamma}\big(I; H^2(\hat{Y}_\infty \setminus \hat{Y}_0)\big). \tag{5.4.26}$$

First we prove that

$$\partial_{\hat{x}} w \in C^{m_\Gamma}\big(I; \mathbb{H}\big(\hat{Y}_\infty\big)\big). \tag{5.4.27}$$

Indeed using same methods than proof of [57, Theorem 4.21]) we get that for all (t) we have $\partial_{\hat{x}} w(t;\cdot) \in \mathbb{H}\big(\hat{Y}_\infty\big)$. Let $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ such that $\partial_{\hat{x}} v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ then taking $v = \partial_{\hat{x}} v$ in

(5.4.24) yields:

$$\begin{aligned}\int_{\hat{Y}_\infty} \partial_{\hat{x}}^2 \hat{\rho} v d\hat{x} d\hat{\nu} &= \int_{\hat{Y}_\infty} \hat{\rho}\big(M(t) \widehat{\nabla}\, w(t;\cdot), \widehat{\nabla}\, \partial_{\hat{x}} v\big) d\hat{x} d\hat{\nu},\\ &= -\int_{\hat{Y}_\infty} \Big(M(t)\partial_{\hat{x}}\big(\hat{\rho}\, \widehat{\nabla}\, w(t;\cdot)\big), \widehat{\nabla}\, v\Big) d\hat{x} d\hat{\nu},\\ &= -\int_{\hat{Y}_\infty} \Big(M(t)\big(\hat{\rho}\, \widehat{\nabla}\, \partial_{\hat{x}} w(t;\cdot)\big), \widehat{\nabla}\, v\Big) d\hat{x} d\hat{\nu}\\ &\quad -\int_{\hat{Y}_\infty} \Big(M(t)\big(\partial_{\hat{x}}\hat{\rho}\big)\, \widehat{\nabla}\, w(t;\cdot)\big), \widehat{\nabla}\, v\Big) d\hat{x} d\hat{\nu}.\end{aligned}$$

Moreover is clear that we have $\int_\Sigma \partial_{\hat{x}} w(t;\cdot) d\hat{x} = 0$ and then we have that for all t, t' the function $\partial_{\hat{x}} w(t;\cdot)$ is given by:

$$\partial_{\hat{x}} w(t;\cdot) = A(t)^{-1} \cdot L'(t), \tag{5.4.28}$$

where we defined the linear form field L' defined for $(t) \in I \times]0,1[$ and $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ by:

$$\langle L'(t), v\rangle_{\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger - \mathbb{H}\left(\hat{Y}_\infty\right)} = \int_{\hat{Y}_\infty} \partial_{\hat{x}}^2 \hat{\rho} v d\hat{x} d\hat{\nu} - \int_{\hat{Y}_\infty} \Big(M(t)\big(\partial_{\hat{x}}\hat{\rho}\big)\, \widehat{\nabla}\, w(t;\cdot)\big), \widehat{\nabla}\, v\Big) d\hat{x} d\hat{\nu}.$$

Moreover thanks to $w \in C^{m_\Gamma}\big(I; \mathbb{H}\big(\hat{Y}_\infty\big)\big)$ we show that $L \in C^{m_\Gamma}\Big(I; \mathbb{H}\big(\hat{Y}_\infty\big)^\dagger\Big)$ and then combining with (5.4.28) end the proof of (5.4.27). Therefore to end the proof of (5.4.26) it is sufficient to proof that:

$$\partial_{\hat{\nu}}^2 w \in C^{m_\Gamma}\big(I; L^2(\hat{Y}_0)\big) \quad \text{and} \quad \partial_{\hat{\nu}}^2 w \in C^{m_\Gamma}\big(I; L^2(\hat{Y}_\infty \setminus \hat{Y}_0)\big). \tag{5.4.29}$$

Indeed we have for all $t \in I$ in the sense of distribution:

$$\partial_{\hat{\nu}} \hat{\rho} \partial_{\hat{\nu}} w(t;\cdot) = \partial_{\hat{x}} \hat{\rho} - \partial_{\hat{x}} \hat{\rho} t' \partial_{\hat{x}} w(t;\cdot).$$

Combining this last identity with $\hat{\rho} \equiv 1$ on $]0,1[\times]0,\infty[$ first yields that:

$$\partial_{\hat{\nu}}^2 w = -t' \partial_{\hat{x}}^2 w \quad \text{on} \quad I \times \Big(]0,1[\times]0,\infty[\Big). \tag{5.4.30}$$

Moreover also combining with the regularity of the function $\hat{\rho}$ in $I \times \Big(]0,1[\times]1,0[\Big)$ yields that we can use Leibniz formula and then we get:

$$\partial_{\hat{\nu}}^2 w = -\big(\partial_{\hat{\nu}} \ln(\hat{\rho})\big) \cdot \partial_{\hat{\nu}} w - t' \hat{\rho}^{-1} \partial_{\hat{x}} \hat{\rho} \hat{\rho} \partial_{\hat{x}} w \quad \text{on} \quad I \times \Big(]0,1[\times]-1,0[\Big). \tag{5.4.31}$$

Moreover thanks to (5.4.27) we have that $\partial_{\hat{x}}^2 w$ and $\partial_{\hat{x}} \hat{\rho} \partial_{\hat{x}} w$ belongs to $C^{m_\Gamma}(I; L^2(\hat{Y}_0))$ and $C^{m_\Gamma}(I; L^2(\hat{Y}_\infty \setminus \hat{Y}_0))$. Thus combining these last properties with (5.4.30) and (5.4.31) end the proof of (5.4.29) and so the the proof of (5.4.26). Thus we finished the proof of whole the proposition. □

We introduce for convenience the operator:

$$\widehat{\nabla} := \begin{pmatrix} \partial_{\hat{x}} \\ \partial_{\hat{\nu}} \end{pmatrix}.$$

Proposition 5.4.10. *There exists $C > 0$ independent of $h, \hat{h}, L$ such that we have the following estimate:*

$$\left\| \widehat{\nabla}\, w - 1_{\hat{\nu}<L} \cdot \widehat{\nabla}\, w_{\hat{h},1/L} \right\|_{C^{m_\Gamma}\left(I;L^2(\hat{Y}_\infty)\right)} \leq C \cdot \left(\hat{h} + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}})\right).$$

Proof. We precise that for whole this proof C designate a generic positive constant which is independent of $\hat{h}$ and L.

We can prove that:

$$\left\| \widehat{\nabla}\, w \right\|_{C^{m_\Gamma}\left(I;L^2(\hat{Y}_\infty \setminus \hat{Y}_L)\right)} \leq C \cdot \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}}), \tag{5.4.32}$$

and then it remains to establish the following estimate to end our proof:

$$\left\| w - w_{\hat{h},1/L} \right\|_{C^{m_\Gamma}\left(I;H^1(\hat{Y}_L)\right)} \leq C \cdot \epsilon, \tag{5.4.33}$$

with $\epsilon := \hat{h} + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}})$. We emphasize that this result is just a simple extension of Cea's lemma(see [28, Theorem 13.1]) . This last estimate exactly means that $P(\alpha)$ is true for $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq m_\Gamma$ where we defined for α the proposition

$$P(\alpha) \Leftrightarrow \left\| \partial_\alpha (w - w_{\hat{h},1/L}) \right\|_{C^0\left(I;H^1(\hat{Y}_L)\right)} \leq C \cdot \epsilon.$$

We prove this result by recurrence on m_Γ in the sense that we show that $P(0)$ is true and for all α with $|\alpha| \leq m_\Gamma$ we have the implication:

$$\left(\forall \alpha' \leq \alpha, \alpha' \neq \alpha,\ P(\alpha')\right) \Rightarrow P(\alpha)$$

Indeed assume that $\alpha = 0$ or for all $\alpha' \leq \alpha$ with $\alpha \neq \alpha$ and prove that $P(\alpha)$ is true. Thanks to the uniform coercivity property of the operator $A_{h,\hat{h},L}$, a sufficient condition is to estimate the following quantity:

$$Q(t) := \left\langle A_{\hat{h},L}(t) E(t), E(t) \right\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)},$$

where we defined $E(t) := \partial_\alpha\big(w(t;\cdot) - w_{\hat{h},1/L}(t;\cdot)\big)$. Thus this last quantities can be rewritten in the following form:

$$Q(t) = \left\langle A_{\hat{h},L}(t) E(t), \partial_\alpha\big(w(t;\cdot) - \Pi_{T_{\hat{h}}} w(t;\cdot)\big) + \partial_\alpha q_h(t) \right\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)},$$

where we defined $q_h := \Pi_{T_{\hat{h}}} w(t;\cdot) - w_{\hat{h},1/L}(t)$ and thanks to Proposition 5.4.23 we get the following estimate:

$$Q(t) \leq Q'(t) + C \cdot \left\| E(t) \right\|_{H^1(\hat{Y}_L)} h, \tag{5.4.34}$$

where we defined the quantity $Q'(t) = \left| \left\langle A_{\hat{h},L}(t) E(t), \partial_\alpha q_h(t) \right\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} \right|$. Classically in the proof of cea's lemma this last quantities vanishes and then we will estimate this last quantity:

$$\begin{aligned} Q'(t) \leq & \left| \left\langle A_{\hat{h},L}(t) w_{\hat{h},1/L}(t;\cdot) - A(t) w(t;\cdot), \mathrm{Ext}_L \partial_\alpha q_h(t) \right\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} \right| \\ & + \left| \left\langle (A_{\hat{h},L} - A) w(t;\cdot), \partial_\alpha \mathrm{Ext}_L q_h(t) \right\rangle_{H^1_\#(\hat{Y}_L)^\dagger - H^1_\#(\hat{Y}_L)} \right|, \end{aligned} \tag{5.4.35}$$

where $\mathrm{Ext}_L : H^1(\hat{Y}_L) \mapsto H^1(\hat{Y}_\infty)$ is a simple extension operator defined for $\phi \in H^1(\hat{Y}_L)$ and $(\hat{x},\hat{\nu}) \in]0,1[\times]L,\infty[$ by $\mathrm{Ext}_L\Big(\phi(\hat{x},\hat{\nu})\Big) = \phi(2\cdot L - \hat{\nu})\chi(\hat{\nu} - L)$ where χ is a smooth cut off function satisfying $\chi = 1$ on $]0,1/2[$ and $\chi = 0$ on $]1/2,1[$ and it is easy to prove that we have existence of $C > 0$ independent of L such that

$$\|\mathrm{Ext}_L\|_{\mathcal{L}\left(H^1(\hat{Y}_L),H^1(\hat{Y}_\infty)\right)} \leq C. \tag{5.4.36}$$

Moreover thanks to Proposition 5.4.23 we get $\|\partial_\alpha q_h\|_{H^1(\hat{Y}_L)} \leq C\Big(\Big\|E(t)\Big\|_{H^1(\hat{Y}_L)} + h\Big)$. Thus combining this last estimate with (5.4.35), (5.4.36) and (5.3.1) yields:

$$Q'(t) \leq C\cdot\Big(Q''(t) + h + \hat{h}\Big)\Big(\Big\|E(t)\Big\|_{H^1(\hat{Y}_L)} + h\Big), \tag{5.4.37}$$

where we defined:

$$Q''(t) := \sup_{v_h\in V_{\#}^{L,\hat{h}}, \|v_h\in V_{\#}^{L,\hat{h}}\|=1} \Big|\Big\langle A_{\hat{h},L}(t)w_{\hat{h},1/L}(t;\cdot) - A(t)w(t;\cdot), \mathrm{Ext}_L\partial_\alpha v_h\Big\rangle_{H^1_{\#}(\hat{Y}_L)^\dagger - H^1_{\#}(\hat{Y}_L)}\Big|.$$

Now let us estimate the quantity $Q''(t)$.

First we extend for convenience the binomial coefficients for $(p,q) \in \Big(\mathbb{N}^q\Big)^2$ for the dimension 2 by:

$$\binom{p}{q} := \prod_{i=1}^{q}\binom{p_i}{q_i},$$

because we have the following useful extension of Leibniz formula for function defined on $\mathbb{R}^q$:

$$\partial_p(fg) = \sum_{q\leq p}\binom{p}{q}\partial_{p-q}f\cdot\partial_q g, \tag{5.4.38}$$

where $q \leq p$ means for all i, $q_i \leq p_i$(The proof of this result is a simple recurrence on the dimension). Combining this last identity with (5.4.24) yields:

$$\begin{cases} \Big\langle A_{\hat{h},L}(t)\cdot\partial_\alpha w_{\hat{h},1/L}(t;\cdot), v_L\Big\rangle_{H^1_{\#}(\hat{Y}_L)^\dagger - H^1_{\#}(\hat{Y}_L)} = Q_1, \\ \langle A(t)\cdot\partial_\alpha w(t;\cdot), v\rangle_{\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger - \mathbb{H}\left(\hat{Y}_\infty\right)} = Q_2, \end{cases}$$

where we defined:

$$\begin{cases} Q_1 := -\displaystyle\sum_{\alpha'\leq\alpha,\alpha'\neq\alpha}\binom{\alpha}{\alpha'}\Big\langle\partial_{\alpha-\alpha'}A_{\hat{h},L}(t)\cdot\partial_{\alpha'}w_{\hat{h},1/L}(t;\cdot), v_L\Big\rangle_{H^1_{\#}(\hat{Y}_L)^\dagger - H^1_{\#}(\hat{Y}_L)} + \\ \qquad \Big\langle\partial_\alpha L_{\hat{h},L}(t), v_L\Big\rangle_{H^1_{\#}(\hat{Y}_L)^\dagger - H^1_{\#}(\hat{Y}_L)}, \\ Q_2 := -\displaystyle\sum_{\alpha'\leq\alpha,\alpha'\neq\alpha}\binom{\alpha}{\alpha'}\langle\partial_{\alpha-\alpha'}A(t)\cdot\partial_{\alpha'}w(t;\cdot), v\rangle_{\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger - \mathbb{H}\left(\hat{Y}_\infty\right)} + \\ \qquad \langle\partial_\alpha L(t), v\rangle_{\mathbb{H}\left(\hat{Y}_\infty\right)^\dagger - \mathbb{H}\left(\hat{Y}_\infty\right)}. \end{cases}$$

Subtracting these two last lines yields for $\alpha \neq 0$:

$$Q^{''}(t) \le Ch + \sum_{\alpha' \le \alpha, \alpha' \neq \alpha} C \sup_{v_h \in V_{\#}^{L,\hat{h}}, \|v_h \in V_{\#}^{L,\hat{h}}\|=1} \langle \beta_{\alpha'}, v_h \rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)} \text{ if } \alpha \neq 0, \quad (5.4.39)$$

where we defined the anti-linear form $\beta_{\alpha'}$ for $v_h \in V_{\#}^{L,\hat{h}}$ by $\langle \beta_{\alpha'}, v_h \rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)} :=$

$$\Big\langle \partial_{\alpha-\alpha'} A_{\hat{h},L}(t) \cdot \partial_{\alpha'} w_{\hat{h},1/L}(t;\cdot) - \partial_{\alpha-\alpha'} A(t) \cdot \partial_{\alpha'} w(t;\cdot), \mathrm{Ext}_L v_h \Big\rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)},$$

and $Q^{''}(t,t) \le Ch$ if $\alpha = 0$ on the one hand. On the other hand thanks to our recurrence hypothesis we get the following estimate for all $\alpha' \le \alpha$ with $\alpha' \neq \alpha$:

$$\begin{aligned}
\Big|\langle \beta_{\alpha'}, v_h \rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)}\Big| \le & \Big|\Big\langle \partial_{\alpha-\alpha'} A_{\hat{h},L}(t) \cdot \big(\partial_{\alpha'} w_{\hat{h},1/L}(t;\cdot) - \partial_{\alpha'} w(t;\cdot)\big), v_h \Big\rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)}\Big| + \\
& \Big|\Big\langle \big(\partial_{\alpha-\alpha'} A_{\hat{h},L}(t) - \partial_{\alpha-\alpha'} A(t)\big) \cdot \partial_{\alpha'} w(t;\cdot), \mathrm{Ext}_L v_h \Big\rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)}\Big|, \\
\le & \underbrace{\Big\|\partial_{\alpha'} w_{\hat{h},1/L}(t) - \partial_{\alpha'} w(t;\cdot)\Big\|}_{\le C\epsilon \text{ by induction hypothesis}} \|v_h\|_{H^1(\hat{Y}_L)} + \\
& \Big|\Big\langle \big(\partial_{\alpha-\alpha'} A_{\hat{h},L}(t) - \partial_{\alpha-\alpha'} A(t)\big) \cdot \partial_{\alpha'} w(t;\cdot), \mathrm{Ext}_L v_h \Big\rangle_{H^1_{\#}(\hat{Y}_L)^{\dagger} - H^1_{\#}(\hat{Y}_L)}\Big|, \\
\le & C\epsilon \|v_h\|_{H^1(\hat{Y}_L)} + \\
& \underbrace{\Big|\int_{\hat{Y}_L} \Big(\partial_{\alpha-\alpha'} \Big(\big(\hat{\rho}_{\hat{h}} - \hat{\rho}\big) M(t)\Big) \cdot \partial_{\alpha'} \widehat{\nabla}\, w(t;\cdot), \widehat{\nabla}\, v_h\Big)\, d\hat{x} d\hat{\nu}\Big|}_{\le C\epsilon \|v_h\|_{H^1(\hat{Y}_L)} \text{ thanks to } (5.3.1)} + \\
& \underbrace{\Big|\int_{\hat{Y}_\infty \setminus \hat{Y}_L} \Big(\big(\partial_{\alpha-\alpha'} M(t)\big) \cdot \partial_{\alpha'} \widehat{\nabla}\, w(t), \widehat{\nabla}\, \mathrm{Ext}_L v_h\Big)\Big|}_{\le C\epsilon \|v_h\|_{H^1(\hat{Y}_L)} \text{ thanks to } (5.4.32)} \le C\epsilon \|v_h\|_{H^1(\hat{Y}_L)},
\end{aligned}$$

which leads combined with (5.4.39) to the estimate $Q^{''}(t) \le C\epsilon$ is also satisfied for $\alpha \neq 0$. Next combining this last estimate with (5.4.34) and (5.4.37) yields:

$$\Big\|E(t)\Big\|^2_{H^1(\hat{Y}_L)} \le C\epsilon\Big(\Big\|E(t)\Big\|_{H^1(\hat{Y}_L)} + \epsilon\Big).$$

Using the Young inequality leads to:

$$\forall \eta > 0,\ 2\|E(t)\|^2_{L^2(\hat{Y}_L)} \le C\eta^{-1}\epsilon^2 + C\eta\Big(h + \|E(t)\|_{L^2(\hat{Y}_L)}\Big)^2 \le C\eta^{-1}\epsilon^2 + 2C\eta\Big(h^2 + \|E(t)\|^2_{L^2(\hat{Y}_L)}\Big),$$

which leads to

$$(2 - 2C\eta) \cdot \|E(t)\|^2_{L^2(\hat{Y}_L)} \le C\Big(\eta^{-1}\epsilon^2 + 2C\eta h^2\Big) \le C\Big(\eta^{-1}\epsilon^2 + 2C\eta\epsilon^2\Big).$$

Now we chose η small enough to have $(2 - 2C\eta) > 0$ and then we have:

$$\|E(t)\|^2_{L^2(\hat{Y}_L)} \le C \cdot (2 - 2C\eta)^{-1} \cdot \Big(\eta^{-1} + 2C\Big)\epsilon^2,$$

which leads to the desired result $\|E(t)\|_{L^2(\hat{Y}_L)} \le C\epsilon$. Therefore we finish our proof. □

Corollary 5.4.11. *There exists $C > 0$ independent of $h, \hat{h}, L$ such that the following estimates hold:*

$$\left\| \boldsymbol{M}_0^\rho - \boldsymbol{M}_{0,\hat{h},1/L}^\rho \right\|_{C^{m_\Gamma}(I)} \leq C \cdot \left(h + \hat{h} + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}}) \right),$$

and

$$\left\| \boldsymbol{M}_1^\rho - \boldsymbol{M}_{1,\hat{h},1/L}^\rho \right\|_{C^{m_\Gamma}(I)} \leq C \cdot \left(h + \hat{h} + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}}) \right).$$

Proof. It is a direct consequence of Proposition 5.4.10 and the expression of the quantity $\boldsymbol{M}_0^\rho$ and $\boldsymbol{M}_1^\rho$. □

Since $m_\Gamma \geq 2$, we directly have the following result:

Proposition 5.4.12. *There exists $C > 0$ independent of $h, \hat{h}, \Delta T, L$ such that the following estimates hold:*

$$\left\| \boldsymbol{M}_0^\rho - I_{\Delta T} \boldsymbol{M}_{0,\hat{h},1/L}^\rho \right\|_{C^{m_\Gamma}(I)} \leq C \cdot \left(h + \hat{h} + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}}) + \Delta T^2 \right),$$

and

$$\left\| \boldsymbol{M}_1^\rho - I_{\Delta T} \boldsymbol{M}_{1,\hat{h},1/L}^\rho \right\|_{C^{m_\Gamma}(I)} \leq C \cdot \left(h + \hat{h} + \exp(-2\pi \cdot L \cdot \sqrt{g_{\min}}) + \Delta T^2 \right).$$

Proposition 5.4.13. *There exists $C > 0$ independent of h such that the following estimate holds:*

$$\left\| g \circ \phi_h^{-1} - g_h \right\|_{L^\infty(\Gamma_h)} + \left\| c \circ \phi_h^{-1} - c_h \right\|_{L^\infty(\Gamma_h)} \leq C \cdot h.$$

Proof. This result is obvious. □

Chapter 6

Numerical approximation of the exact solution

Here, we present an error estimate for the approximation of the exact solution. Since the exact solution have fast oscillations near the boundary Γ, we will use a mesh which is finer near the boundary Γ and small compared to the small parameter. This notion of "refinement" is mathematically written in (6.5.18). Theorem 6.6.1 gives an error estimate in function of the small parameter δ, the size of mesh near the boundary and the size of mesh far from the boundary Γ. The size of mesh near the boundary has to be small compared to the small parameter δ. This explain why the approximate model is more efficient than the exact one.

6.1 The mesh of the domain Ω^δ

Let $n_h : \Gamma_h \mapsto \mathbb{R}^3$ be an approximation of the normal unit n in the sense that we have existence of $C > 0$ such that for all $h > 0$ the followings estimate holds:

$$\|n_h \circ \phi_h - n\|_{W^{1,\infty}(\Gamma)} \leq C \cdot h \quad \text{and} \quad \|n_h \circ \phi_h - n\|_{W^{0,\infty}(\Gamma)} \leq C \cdot h^2. \tag{6.1.1}$$

From this last approximation of the normal unit we build an approximation of the surface Γ_δ defined by the unique polygonal surface whose corner are:

$$\left\{\Gamma^h_{i,\delta} := \Gamma_i - \delta \cdot n_h(\Gamma_i)\right\}_i.$$

Hence the polygon Ω^h_δ is given by the unique open set whose boundary is $\Gamma^h_\delta \cup \partial\mathbb{B}^{1/3}_h$. (See Figure 6.1).

6.2 Finite element method

Let V^δ_h be the space of discretization P_1 on the mesh T^h_δ and let ρ^δ_h and μ^δ_h defined by 1 in Ω_h and defined by the P_1 interpolation in the numerical thin coat $\Omega^h_\delta \setminus \Omega_h$. Then the function u^δ_h by the unique solution of: Find $u^\delta_h \in V^\delta_h$ such that for all $v^\delta_h \in V^\delta_h$ we have:

$$\int_{\Omega^h_\delta} \rho^\delta_h(\nabla u^\delta_h, \nabla v^\delta_h) - k^2 \mu^\delta_h u^\delta_h \overline{v}^\delta_h d\Omega^\delta_h = \int_\Omega f \circ \phi_h d\Omega. \tag{6.2.2}$$

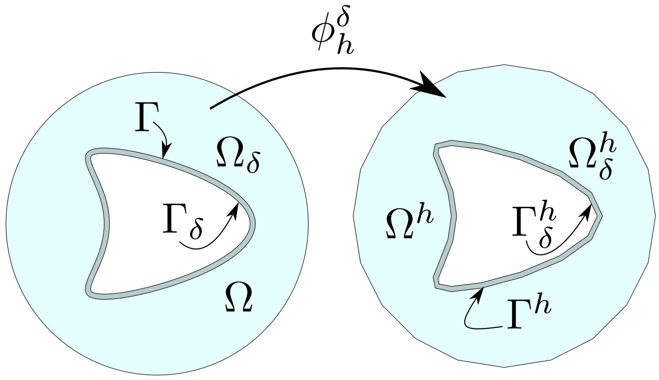


Figure 6.1: Illustration of the ϕ_δ^h application

6.3 Construction of a map $\phi_h^\delta : \Omega^\delta \mapsto \Omega_h^\delta$

Since we have $\Omega^\delta \neq \Omega_h^\delta$ we need prove the following extension of Proposition 5.4.1:
Proposition 6.3.1. *For all $\delta, h > 0$ there exists a bijective application $\phi_h^\delta : \Omega_\delta^h \mapsto \Omega$ such that we have*

$$\phi_h^\delta(\Gamma_\delta) = \Gamma_\delta^h \quad \text{and} \quad \phi_h^\delta(\Gamma) = \Gamma^h,$$

and we have

$$\phi_h^\delta(\Omega_\delta) = \Omega_\delta^h \quad \text{and} \quad \phi_h^\delta(\Omega) = \Omega^h.$$

Moreover there exists $C > 0$ independent of δ and h such that the following estimate holds:

$$\|\mathbb{I} - \phi_h^\delta\|_{W^{0,\infty}(\Omega_\delta)} \leq C \cdot h^2 \quad \text{and} \quad \|\mathbb{I} - \phi_h^\delta\|_{W^{1,\infty}(\Omega_\delta)} \leq C \cdot h. \tag{6.3.3}$$

Proof. First we introduce an intermediate function $\tilde{\phi}_h$ defined on Ω_δ. Let χ be a smooth cut-off function such that $\chi(x) = 0$ for $x \leq_1$ and $\chi = 1$ for $x \geq 0$ and this function is defined for $x = x_\Gamma + \nu n(x_\Gamma)$ by:

$$\tilde{\phi}_h(x) := x + \chi\left(\frac{\nu}{h}\right)\left(\phi_h(x_\Gamma) - x_\Gamma + (n_h(x_\Gamma) - n(x_\Gamma)) \cdot \nu\right),$$

where ϕ_h is the function appearing in Proposition 5.4.1. We introduce this last function because this last one transforms the corner of the triangulation of Γ into the corner of Γ_δ i.e.

$$\forall i,\ \phi_h\left(\Gamma_i - \delta n(\Gamma_i)\right) = \Gamma_{i,\delta}^h,$$

and we easy have: $\phi_h(\Gamma) = \Gamma_h$. Moreover we will later prove that this function satisfies similar to the required (6.3.3):

$$\left\|\tilde{\phi}_h - \mathbb{I}\right\|_{W^{0,\infty}(\Omega_\delta)} \leq C \cdot h^2 \quad \text{and} \quad \left\|\tilde{\phi}_h - \mathbb{I}\right\|_{W^{1,\infty}(\Omega_\delta)} \leq C \cdot h. \tag{6.3.4}$$

Unfortunately this function does not not transform the set Γ_δ into a polygon i.e.

$$\tilde{\phi}_h(\Gamma_\delta) \neq \Gamma_\delta^h.$$

However using Proposition 5.4.1 with $\Gamma = \tilde{\phi}_h(\Gamma_\delta)$ yields existence of a function $\tilde{\Phi}_h^\delta$: $\tilde{\phi}_h(\Omega_\delta) \mapsto \Omega_\delta^h$ satisfying existence of $C > 0$ independent of h, δ such that:

$$\left\|\tilde{\Phi}_h^\delta - \mathbb{I}\right\|_{W^{0,\infty}(\tilde{\phi}_h(\Omega_\delta))} \leq C \cdot h^2 \quad \text{and} \quad \left\|\tilde{\Phi}_h^\delta - \mathbb{I}\right\|_{W^{1,\infty}(\tilde{\phi}_h(\Omega_\delta))} \leq C \cdot h, \tag{6.3.5}$$

and we have the following property:

$$\tilde{\Phi}_h^\delta\big(\tilde{\phi}_h(\Gamma_\delta)\big) = \Gamma_\delta^h \quad \text{and} \quad \tilde{\Phi}_h^\delta\left(\tilde{\phi}_h(\Gamma)\right) = \Gamma^h.$$

Therefore now we can define the application ϕ_h^δ by:

$$\phi_h^\delta := \tilde{\Phi}_h^\delta \circ \tilde{\phi}_h^\delta,$$

and this last application well satisfies the desired property (6.3.3). Indeed on the one hand we have for all $x \in \Omega_\delta$:

$$\left|\left(\tilde{\Phi}_h^\delta \circ \tilde{\phi}_h^\delta\right)(x) - x\right| = \left|\tilde{\Phi}_h^\delta(\tilde{\phi}_h^\delta(x)) - x\right| \leq \left|\tilde{\Phi}_h^\delta(\tilde{\phi}_h^\delta(x)) - \tilde{\phi}_h^\delta(x)\right| + \left|\tilde{\phi}_h^\delta(x) - x\right| \leq 2C \cdot h^2,$$

which leads combined with (6.3.4) and (6.3.5) to:

$$\left\|\tilde{\Phi}_h^\delta \circ \tilde{\phi}_h^\delta - \mathbb{I}\right\|_{W^{0,\infty}(\Omega_\delta)} \leq 2C \cdot h^2. \tag{6.3.6}$$

On the other hand we have for all $x \in \Omega_\delta$:

$$\begin{aligned}
\left|\mathrm{D}\left(\tilde{\Phi}_h^\delta \circ \tilde{\phi}_h^\delta\right)(x) - \mathbb{I}\right| &= \left|\mathrm{D}\,\tilde{\Phi}_h^\delta(\tilde{\phi}_h^\delta(x)) \cdot \mathrm{D}\,\tilde{\phi}_h^\delta(x) - \mathbb{I}\right| \\
&\leq \left|\mathrm{D}\,\tilde{\Phi}_h^\delta(\tilde{\phi}_h^\delta(x)) \cdot \mathrm{D}\,\tilde{\phi}_h^\delta(x) - \mathrm{D}\,\tilde{\phi}_h^\delta(x)\right| + \left|\mathrm{D}\,\tilde{\phi}_h^\delta(x) - \mathbb{I}\right| \\
&\leq \left|\mathrm{D}\,\tilde{\Phi}_h^\delta(\tilde{\phi}_h^\delta(x)) - \mathbb{I}\right| \cdot \left|\mathrm{D}\,\tilde{\phi}_h^\delta(x)\right| + \left|\mathrm{D}\,\tilde{\phi}_h^\delta(x) - \mathbb{I}\right| \leq 2C \cdot h,
\end{aligned}$$

which leads combined with (6.3.4) and (6.3.5) to:

$$\left\|\tilde{\Phi}_h^\delta \circ \tilde{\phi}_h^\delta - \mathbb{I}\right\|_{W^{1,\infty}(\Omega_\delta)} \leq 2C \cdot h.$$

Thus combining this last estimate with the estimate (6.3.6) and the invertibility of $\tilde{\Phi}_h^\delta$ and $\tilde{\phi}_h^\delta$ ends the proof of (6.3.3). Therefore it remains to prove estimate (6.3.4). Indeed we have the following decomposition:

$$\tilde{\phi}_h - \mathbb{I} = \Delta_h \circ \mathcal{L}, \tag{6.3.7}$$

where $\Delta_h : \Gamma\times]-\delta, \eta_0[$ is defined for $(x_\Gamma, \nu) \in \Gamma\times]-\delta, \eta_0[$ by:

$$\Delta_h(x_\Gamma, \nu) := \chi\left(\frac{\nu}{h}\right)\left(\phi_h(x_\Gamma) - x_\Gamma + (n_h(x_\Gamma) - n(x_\Gamma)) \cdot \nu\right).$$

Thus using Leibniz formula yields:

$$\begin{aligned}
\|\Delta_h\|_{W^{1,\infty}(\Gamma\times]-\delta,\eta_0[)} \leq & \left\|\chi\left(\frac{\nu}{h}\right)\right\|_{W^{1,\infty}(\Gamma\times]-\delta,\eta_0[)} \cdot \|\phi_h(x_\Gamma) - x_\Gamma + (n_h(x_\Gamma) - n(x_\Gamma)) \cdot \nu\|_{W^{0,\infty}(\Gamma\times]-\delta,\eta_0[)} + \\
& \left\|\chi\left(\frac{\nu}{h}\right)\right\|_{W^{0,\infty}(\Gamma\times]-\delta,\eta_0[)} \cdot \|\phi_h(x_\Gamma) - x_\Gamma + (n_h(x_\Gamma) - n(x_\Gamma)) \cdot \nu\|_{W^{1,\infty}(\Gamma\times]-\delta,\eta_0[)}, \\
\leq & Ch^{-1}\|\phi_h(x_\Gamma) - x_\Gamma + (n_h(x_\Gamma) - n(x_\Gamma)) \cdot \nu\|_{W^{0,\infty}(\Gamma\times]-\delta,\eta_0[)} + \\
& C\|\phi_h(x_\Gamma) - x_\Gamma + (n_h(x_\Gamma) - n(x_\Gamma)) \cdot \nu\|_{W^{1,\infty}(\Gamma\times]-\delta,\eta_0[)} \\
\leq & Ch^{-1}\Big(\|\phi_h - \mathbb{I}\|_{W^{0,\infty}(\Gamma)} + \|(n_h - n)\|_{W^{0,\infty}(\Gamma)}\Big) + \\
& C\Big(\|\phi_h - \mathbb{I}\|_{W^{1,\infty}(\Gamma)} + \|(n_h - n)\|_{W^{1,\infty}(\Gamma)}\Big).
\end{aligned}$$

Thus thanks to Proposition 5.4.1 and the assumption (6.1.1) this last estimate becomes:

$$\|\Delta_h\|_{W^{1,\infty}(\Gamma\times]-\delta,\eta_0[)} \leq C\cdot h.$$

Moreover thanks to Proposition 5.4.1 and the assumption (6.1.1) we get:

$$\|\Delta_h\|_{W^{1,\infty}(\Gamma\times]-\delta,\eta_0[)} \leq C\cdot h^2.$$

Thus combining these two last estimates with the regularity of the function $\mathcal{L}$ and the decomposition (6.3.7) end the proof of (6.3.4) and so the proof is finished. $\square$

6.4 Explosion of the H^s norm of the function u_δ

Unfortunately the exact solution have the following explosion behavior of its H^s norm for $s>1$:

Proposition 6.4.1. *If the function $\hat{\rho}$ and $\hat{\mu}$ belongs to $C^{m_\Gamma}(\Gamma\times[0,1]^2)$ then for all $q\leq m_\Gamma$ and $\delta>0$ we have $u_\delta\in H^q(C_\delta)\cap H^q(\Omega)$ and there exists $C>0$ independent of δ such that:*

$$\|u^\delta\|_{H^q(\Omega_\delta)} \leq C\delta^{-(q-1)}. \tag{6.4.8}$$

Proof. The proof of $u_\delta\in H^q(C_\delta)\cap H^q(\Omega)$ is already made in [57, Theorem 4.21]. Therefore it is sufficient to prove the estimate (6.4.8)

By using charts and unit partition of unity, we can assume that Γ is $\mathbb{R}\times\{0\}$ and $\Omega_\delta=\mathbb{R}\times]-\delta,1[$. Thanks to Leibniz formula we have:

$$\operatorname{div}\left(\rho^\delta\nabla\partial_y u^\delta\right)=g^\delta \quad\text{and}\quad \partial_\nu\partial_y u^\delta=0 \text{ on } \partial\Omega_\delta, \tag{6.4.9}$$

with $g^\delta:=\partial_y f-\operatorname{div}(\partial_y\rho^\delta)\nabla\partial_y u^\delta-k^2\partial_y(\mu^\delta u^\delta)$ and it is easy to prove existence of $C>0$ independent of δ such that the following estimate holds:

$$\|g^\delta\|_{(H^1(\Omega_\delta))^\dagger} \leq C\delta^{-1},$$

and thanks to (6.4.9) we get existence of $C>0$ independent of δ such that we have $\|\partial_y^q u^\delta\|_{H^1(\Omega_\delta)}\leq C\delta^{-1}$. We can prove by recurrence that in fact we can extend this last estimate in the sense that there exists $C>0$ independent of δ such that for all $q\leq m_\Gamma-1$ we have:

$$b\|\partial_y^q u^\delta\|_{H^1(\Omega_\delta)} \leq C\delta^{-q}. \tag{6.4.10}$$

Now let us prove by recurrence that we have for all $1\leq q\leq m_\Gamma$ existence of $C>0$ independent of δ such that:

$$\left\|\partial_y^{q-l}\partial_x^l u_\delta\right\|_{L^2(C_\delta)}+\left\|\partial_y^{q-l}\partial_x^l u_\delta\right\|_{L^2(\Omega)} \leq C\delta^{-(q-1)}. \tag{6.4.11}$$

Thanks to (6.4.10) we first get the initialization $q=1$. Indeed thanks to this estimate we get existence of $C>0$ such that:

$$\|\partial_y^1 u^\delta\|_{L^2(\Omega_\delta)}+\|\partial_x\partial_y^{1-1}u^\delta\|_{L^2(\Omega_\delta)} \leq C\|u^\delta\|_{H^1(\Omega_\delta)} \leq C\delta^{-(q-1)}.$$

Now let $q\geq 2$ such that we have for all $q'\leq q-1$ that

$$\forall l\leq q',\ \left\|\partial_y^{q'-l}\partial_x^l u_\delta\right\|_{L^2(C_\delta)}+\left\|\partial_y^{q'-l}\partial_x^l u_\delta\right\|_{L^2(\Omega)} \leq C\delta^{-(q'-1)}, \tag{6.4.12}$$

and let us prove that this last estimate implies:

$$\forall l \leq q,\ \left\|\partial_y^{q-l}\partial_x^l u_\delta\right\|_{L^2(C_\delta)} + \left\|\partial_y^{q-l}\partial_x^l u_\delta\right\|_{L^2(\Omega)} \leq C\delta^{-(q-1)}. \tag{6.4.13}$$

Inside this recurrence we do a second recurrence for a fixed $q' \leq$ on $0 \leq l \leq q'$. First let us study the case of $l =$ and $l = 1$. Indeed thanks to (6.4.10) we get

$$\|\partial_y^q u^\delta\|_{L^2(\Omega_\delta)} + \|\partial_x\partial_y^{q-1}u^\delta\|_{L^2(\Omega_\delta)} \leq C\|\partial_y^{q-1}u^\delta\|_{H^1(\Omega_\delta)} \leq C\delta^{-(q-1)}.$$

Therefore it remains to prove that for all $0 \leq l \leq q-2$ we have the following implication:

$$\forall l' \leq l,\ \left\|\partial_y^{q-l'}\partial_x^{l'}u^\delta\right\| \leq C\delta^{-(q-1)} \Rightarrow \left\|\partial_y^{q-(l+2)}\partial_x^{l+2}u^\delta\right\| \leq C\delta^{-(q-1)} \tag{6.4.14}$$

Indeed assume that we have:

$$\forall l' \leq l,\ \left\|\partial_y^{q-l'}\partial_x^{l'}u^\delta\right\| \leq C\delta^{-(q-1)}. \tag{6.4.15}$$

We have from the equation satisfied by our solution u^δ: $\partial_x\rho^\delta\partial_x u^\delta + \partial_y\rho^\delta\partial_y u^\delta + k^2\mu^\delta u^\delta = f$ the following useful equality:

$$\partial_x^2 u^\delta = A^\delta\partial_y^2 u^\delta + \delta^{-1}B^\delta \cdot \nabla u^\delta + D^\delta u^\delta + (g^\delta)^{-1}f,$$

where A^δ, B^δ and C^δ are function $\psi_\Gamma-\delta-$periodic, which leads to the following useful identity:

$$\begin{aligned}\partial_y^{q-(l+2)}\partial_x^{l+2}u^\delta &= \partial_y^{q-(l+2)}\partial_x^l\left(\partial_x^2 u^\delta\right), \\ &= \partial_y^{q-(l+2)}\partial_x^l\left(A^\delta\partial_y^2u^\delta + \delta^{-1}B^\delta\cdot\nabla u^\delta + D^\delta u^\delta + (g^\delta)^{-1}f\right).\end{aligned}$$

Moreover combining with the identity (5.4.38) yields existence of $C > 0$ independent of δ such that we have:

$$\left\|\partial_y^{q-(l+2)}\partial_x^{l+2}u^\delta\right\| \leq C \sum_{\alpha' \leq q-(l+2)\alpha \leq l}\left(Q_a^{\alpha',\alpha} + \delta^{-1}Q_b^{\alpha',\alpha} + Q_d^{\alpha',\alpha} + \delta^{-(q-2)}\right) \tag{6.4.16}$$

where we defined the following quantities:

$$\begin{cases} Q_a^{\alpha',\alpha} = \left\|\partial_y^{q-(l+2)-\alpha'}\partial_x^{l-\alpha}A^\delta\right\| \times \left\|\partial_y^{2+\alpha'}\partial_x^\alpha u^\delta\right\|, \\ Q_b^{\alpha',\alpha} = \left\|\partial_y^{q-(l+2)-\alpha'}\partial_x^{l-\alpha}B^\delta\right\| \times \left\|\partial_y^{\alpha'}\partial_x^\alpha\nabla u^\delta\right\|, \\ Q_d^{\alpha',\alpha} = \left\|\partial_y^{q-(l+2)-\alpha'}\partial_x^{l-\alpha}D^\delta\right\| \times \left\|\partial_y^{\alpha'}\partial_x^\alpha u^\delta\right\|. \end{cases}$$

Moreover using that $\{A^\delta, B^\delta, D^\delta\}_\delta$ are $\psi_\Gamma-\delta$-periodic sequences implies that these last estimates become:

$$\begin{cases} Q_a^{\alpha',\alpha} \leq C\cdot\delta^{-q+(\alpha+\alpha'+2)} \times \left\|\partial_y^{2+\alpha'}\partial_x^\alpha u^\delta\right\|, \\ Q_b^{\alpha',\alpha} \leq C\cdot\delta^{-q+(\alpha+\alpha'+2)} \times \left(\left\|\partial_y^{\alpha'+1}\partial_x^\alpha u^\delta\right\| + \left\|\partial_y^{\alpha'}\partial_x^{\alpha+1}u^\delta\right\|\right), \\ Q_d^{\alpha',\alpha} \leq C\cdot\delta^{-q+(\alpha+\alpha'+2)} \times \left\|\partial_y^{\alpha'}\partial_x^\alpha u^\delta\right\|, \end{cases}$$

which also can be rewritten in the following forms:

$$\begin{cases} Q_a^{\alpha',\alpha} \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \left\|\partial_y^{(2+\alpha'+\alpha)-\alpha}\partial_x^{\alpha}u^{\delta}\right\|, \\ Q_b^{\alpha',\alpha} \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \left(\left\|\partial_y^{(\alpha'+\alpha+1)-\alpha}\partial_x^{\alpha}u^{\delta}\right\| + \left\|\partial_y^{(\alpha'+\alpha+1)-(\alpha+1)}\partial_x^{\alpha+1}u^{\delta}\right\|\right), \\ Q_d^{\alpha',\alpha} \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \left\|\partial_y^{(\alpha+\alpha')-\alpha'}\partial_x^{\alpha}u^{\delta}\right\|, \end{cases}$$

Thanks to our recurrence hypothesis (6.4.12) and (6.4.15) these last estimates become:

$$\begin{cases} Q_a^{\alpha',\alpha} \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \delta^{-(2+\alpha+\alpha'-1)} \leq C\delta^{-(q-1)}, \\ Q_b^{\alpha',\alpha} \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \left(\left\|\partial_y^{(\alpha'+\alpha+1)-\alpha}\partial_x^{\alpha}u^{\delta}\right\| + \left\|\partial_y^{(\alpha'+\alpha+1)-(\alpha+1)}\partial_x^{\alpha+1}u^{\delta}\right\|\right), \\ \qquad\quad \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \delta^{-(\alpha+\alpha'+1-1)} \leq C\delta^{-(q-2)}, \\ Q_d^{\alpha',\alpha} \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \left\|\partial_y^{(\alpha+\alpha')-\alpha'}\partial_x^{\alpha}u^{\delta}\right\| \leq C \cdot \delta^{-q+(\alpha+\alpha'+2)} \times \delta^{-(\alpha+\alpha')} \leq C\delta^{(q-1)}, \end{cases}$$

which leads to $Q_a^{\alpha',\alpha} + \delta^{-1}Q_b^{\alpha',\alpha} + Q_d^{\alpha',\alpha} \leq C\delta^{-(q-1)}$. Thus combing this last estimate with (6.4.16) end the proof of the implication (6.4.14). Therefore we succeed (6.4.13), which concludes the proof the induction step of the proof of (6.4.11). Therefore we can conclude. □

Unfortunately the classical approach require the following very restrictive condition: Existence of $C > 0$ such that for all $\delta, h > 0$ we have for all $h_T \in T_h$:

$$h_T \leq C\delta \cdot h. \tag{6.4.17}$$

6.5 Technique of mesh refinement near the boundary Γ

It is however possible to replace the very restrictive condition (6.4.17) by a less restrictive one which is existence of $C > 0$ such that for all δ and $h > 0$ (see Figure 6.2)

$$h_T \leq C\Big(\delta + \operatorname{dist}(T_h, \Gamma)\Big) \cdot h, \tag{6.5.18}$$

where we defined for all subset $\omega \subset \Omega$:

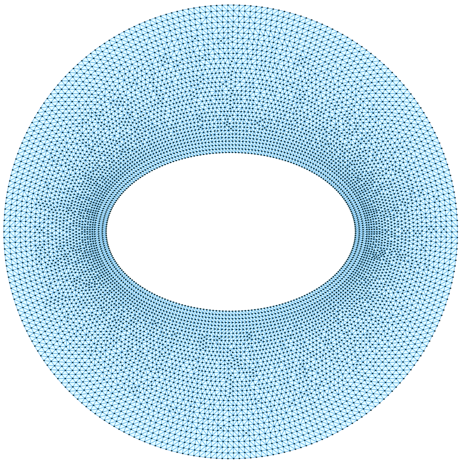
$$\operatorname{dist}(\omega, \Gamma) := \inf_{x \in \omega} \operatorname{dist}(x, \Gamma).$$

Indeed we succeed to prove the following result:

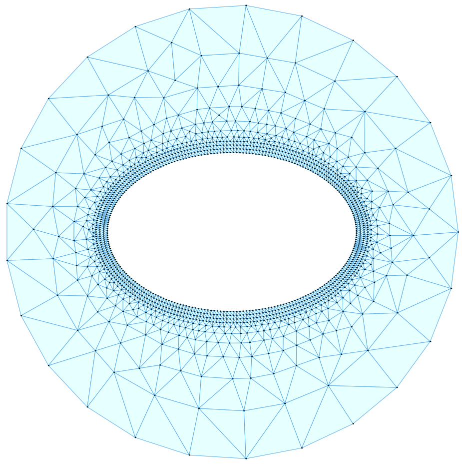
We are inspired by the famous result of interior regularity for the Laplacian operator:

$$\Delta u = 0 \text{ in } \Omega \Rightarrow u \in C^{\infty}(\Omega). \tag{6.5.19}$$

Let us prove the following extension of this result of interior regularity (that can quantify how the regularity of the solution decreases when we approach the boundary Γ.)



(a) Brute force mesh



(b) Mesh with refinement method

Figure 6.2: Mesh refinement

Proposition 6.5.1. *Let $u \in H^1(C_{0,\eta_0})$ and $s \leq m_\Gamma - 2$ such that:*

$$\Delta u + k^2 u = f \in H^s(C_{0,\eta_0}) \quad \text{and} \quad \partial_\nu u = g \in H^{s-\frac{1}{2}}(\Sigma_{\eta_0}),$$

and let $\tilde{\nu} : C_{0,\eta_0} \mapsto \mathbb{R}$ defined for $x \in C_{0,\eta_0}$ by:

$$\tilde{\nu}(x) := \operatorname{dist}(x, \Gamma).$$

Then we have $\tilde{\nu}^{s+1} u \in H^{s+2}(C_{0,\eta_0})$ with existence of $C > 0$ such that:

$$\|\tilde{\nu}^{s+1} u\|_{H^{s+2}(C_{0,\eta_0})} \leq C \left(\|f\|_{H^s(C_{0,\eta_0})} + \|g\|_{H^{s-\frac{1}{2}}(\Sigma_{\eta_0})} \right), \tag{6.5.20}$$

and for all open subsets $\omega \subset C_{0,\eta_0}$, we have the following bound of the H^{s+2} norm of u when ω approachs the boundary Γ:

$$\|u\|_{H^{s+2}(\omega)} \leq C \cdot \sup_{s' \leq s} \|\tilde{\nu}^{s'+1} u\|_{H^{s'+2}(\omega)} \cdot \operatorname{dist}(\omega, \Gamma)^{-(s+1)}. \tag{6.5.21}$$

Let us give somme comment about this result. Indeed (6.5.19) states that the function u appearing in this belongs to $C^\infty(\Omega)$. Thanks to the sobolev embedding theorem, this is equivalent to: For all open bounded $\omega \subset \Omega$ satisfying $\operatorname{dist}(\omega, \Gamma) > 0$, we have $u \in H^{s+2}(\omega)$ for all $s \geq 0$.

However, the restriction u_Γ of u on Γ might be discontinuous. Figure 6.3 is a graphical illustration of a numerical approximation of the unique solution of $\Delta u = 0$ satisfaying $u = 0$ on Σ_{η_0} and for all $(x, y) \in \Gamma$

$$u(x, y) = 1 \text{ if } xy > 0 \quad \text{and} \quad u(x, y) = -1 \text{ else.}$$

Then in this case we a priori have:

$$\lim_{\operatorname{dist}(w,\Gamma) \to 0} \|u\|_{H^s(\omega)} = \infty.$$

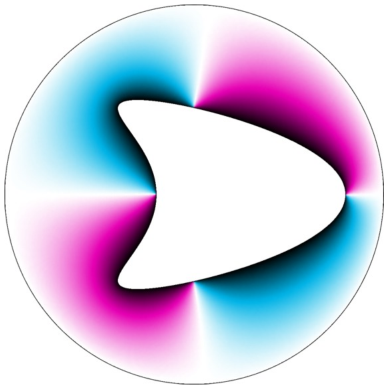


Figure 6.3: Illustration of a function satisfaying $\Delta u = 0$ discontinuous on Γ.

Hence the estimate (6.5.21) appearing Proposition 6.5.1 give an estimate of how the quantity $\|u\|_{H^s(\omega)}$ tends to infinty when $\mathrm{dist}(w, \Gamma) \to 0$.

A second commentary is that we will strongly use this last result in the proof of Proposition 6.6.1. More precisely we use it to estimate the error u^δ and its P^1 interpolation on each cell of the mesh. We recall that for each cell T_h of the mesh, one has:

$$\|u^\delta - \Pi u^\delta\|_{H^1(T_h)} \leq C\|u^\delta\|_{H^2(T_h)} h_T,$$

where Π is the P^1 interpolator and h_T is the diameter of the cell T_h.
Proof of Proposition 6.5.1. Thanks to results of regularity we can assume that $f = 0$ and $g = 0$. First we prove by recurrence that for all $-1 \leq s' \leq s$ we have $\tilde{\nu}^{s'+1} u \in H^{s'+2}(C_{0,\eta_0})$ with existence of $C > 0$ independent of u such that:

$$\|\tilde{\nu}^{s'+1} u\|_{H^{s'+2}(C_{0,\eta_0})} \leq C\|u\|_{H^1(C_{0,\eta_0})}.$$

The result is clear for $s' = -1$ and let $s' < s$ such that this last property is verified. Thanks to Proposition 1.3.1 (See Chapter 1), we get that the function $\tilde{\nu}$ is at least C^2 and u is C^∞ on the interior which leads that we strongly have in $C_{0,\delta}$:

$$\begin{aligned}(\Delta + k^2)\left(\tilde{\nu}^{s'+2} u\right) &= \cancel{\tilde{\nu}^{s'+2}\left(\Delta u + k^2 u\right)} + 2\nabla\tilde{\nu}^{s'+2}\cdot\nabla u + u\Delta\tilde{\nu}^{s'+2},\\ &= 2(s'+2)\tilde{\nu}^{s'+1}\cdot\nabla\tilde{\nu}\cdot\nabla u + u\cdot\mathrm{div}\left((s'+2)\tilde{\nu}^{s'+1}\nabla\tilde{\nu}\right),\\ &= (s'+2)\left(2\tilde{\nu}^{s'+1}\cdot\nabla\tilde{\nu}\cdot\nabla u + u\cdot\tilde{\nu}^{s'+1}\Delta\tilde{\nu} + (s'+1)\tilde{\nu}^{s'}|\nabla\tilde{\nu}|^2 u\right)\end{aligned}$$

Combining this last identity with $|\nabla\tilde{\nu}| = 1$ leads to:

$$(\Delta + k^2)\left(\tilde{\nu}^{s'+2} u\right) = (s'+2)\left(2\tilde{\nu}^{s'+1}\cdot\nabla\tilde{\nu}\cdot\nabla u + u\cdot\tilde{\nu}^{s'+1}\Delta\tilde{\nu} + (s'+1)\tilde{\nu}^{s'} u\right).$$

Moreover we have:

$$\nabla\tilde{\nu}\cdot\tilde{\nu}^{s'+1}\nabla u = \nabla\tilde{\nu}\cdot\nabla(\tilde{\nu}^{s'+1} u) - (s'+1)\tilde{\nu}^{s'} u,$$

which leads to the following identity:

$$(\Delta + k^2)\left(\tilde{\nu}^{s'+2} u\right) = (s'+2)\left(2\cdot\nabla\tilde{\nu}\cdot\nabla(\tilde{\nu}^{s'+1} u) + u\tilde{\nu}^{s'+1}\Delta\tilde{\nu} - (s'+1)\tilde{\nu}^{s'} u\right)$$

Combining our recurrence hypothesis with $\tilde{\nu} \in W^{s'+2}(C_{0,\delta})$ yields existence of $C > 0$ such that:

$$\nabla(\tilde{\nu}^{s'+1}u) \in H^{s'+1}(C_{0,\eta_0}) \quad \text{with} \quad \left\|\nabla(\tilde{\nu}^{s'+1}u)\right\|_{H^{s'+1}(C_{0,\eta_0})} \leq C\|u\|_{H^1(C_{0,\eta_0})},$$

and $u\tilde{\nu}^{s'+1}\Delta\tilde{\nu} - (s'+1)\tilde{\nu}^{s'}u \in H^{s'+1}(C_{0,\eta_0})$ with

$$\left\|u\tilde{\nu}^{s'+1}\Delta\tilde{\nu} - (s'+1)\tilde{\nu}^{s'}u\right\|_{H^{s'+1}(C_{0,\eta_0})} \leq C\|u\|_{H^1(C_{0,\eta_0})},$$

yields existence of C independent of u such that:

$$(\Delta + k^2)\left(\tilde{\nu}^{s'+2}u\right) \in H^{s'+2}(C_{0,\delta}) \quad \text{with} \quad \left\|(\Delta + k^2)\left(\tilde{\nu}^{s'+2}u\right)\right\|_{H^{s'+2}(C_{0,\delta})} \leq C\|u\|_{H^1(C_{0,\delta})}.$$

Since the function $\tilde{\nu}^{s'+2}u$ satisfies the homogeneous Dirichlet condition on Γ then regularity result for the laplacian operator yields this last one belongs to $H^{s'+3}$ with existence of $C > 0$ such that:

$$\left\|\tilde{\nu}^{s'+2}u\right\|_{H^{s'+3}(C_{0,\delta})} \leq C\|u\|_{H^1(C_{0,\delta})}.$$

Thus to finish the proof of this proposition it remains to prove estimate (6.5.21). First let us prove this last result when Γ is the plane $\mathbb{R}^2 \times \{0\}$ because the expression of the function $\tilde{\nu}$ is explicit. Indeed in this case we have for all $(x,\nu) \in \mathbb{R}^2\times]0,\eta_0[$ that $\tilde{\nu}(x,\nu) = \nu$. Thus in this plane case the estimate (6.5.21) is a direct consequence of the following result: For all $q \in \mathbb{N}$ there exists $C_q > 0$ such that for all u smooth enough we have:

$$\left|\partial_\nu^{q+1}u\right| \leq C_q\nu^{-q} \cdot \left(\max_{0\leq q'\leq q}\left|\partial_\nu^{q'+1}(\nu^{q'}u)\right|\right). \tag{6.5.22}$$

Let us prove this last result by recurrence. Indeed this result is clear for $q = 0$ and then assume that the result is true for all $0 \leq q' \leq q$ and let us prove that the result is true for $q + 1$. Thanks to Leibniz formula we have:

$$\partial_\nu^{q+2}(\nu^{q+1} \cdot u) = \sum_{q'=1}^{q+2}\binom{q+2}{q}\partial_\nu^{q'}u \cdot \nu^{q'-1}\frac{(q+1)!}{(q'-1)!},$$

which leads to:

$$\nu^{q+1}\partial_\nu^{q+2}u = \partial_\nu^{q+2}(\nu^{q+1} \cdot u) + \sum_{q'=1}^{q+1}C_{qq'}\nu^{q'-1}\partial_\nu^{q'}u,$$

where we posed for $1 \leq q' \leq q+1$ the quantity:

$$C_{qq'} := -\binom{q+2}{q} \cdot \nu^{q'-1}\frac{(q+1)!}{(q'-1)!}.$$

Thus combining this last identity with our recurrence hypothesis yields:

$$\begin{aligned}\nu^{q+1}|\partial_\nu^{q+2}u| &\leq |\partial_\nu^{q+2}(\nu^{q+1} \cdot u)| + \left(\sum_{q'=1}^{q+1}C_{qq'}\right) \cdot C_q\left(\max_{0\leq q'\leq q}\left|\partial_\nu^{q'+1}(\nu^{q'}u)\right|\right),\\ &\leq \underbrace{\left(1 + C_q\sum_{q'=1}^{q+1}C_{qq'}\right)}_{:=C_{q+1}} \times \max_{0\leq q'\leq q+1}\left|\partial_\nu^{q'+1}(\nu^{q'}u)\right|,\end{aligned}$$

which end the proof of (6.5.22). Now assume that Γ is not the plane and bring us back to the plane case. To do that, first prove existence of C^{m_Γ} partition of unity of C_{0,η_0} and collection of diffeomorphism $\phi_i : w_i \mapsto w_i^\Gamma \times]0,\eta_0[$ where w_i is the interior of the support of the function χ_i and w_i^Γ is an open bounded subset of $\mathbb{R}^2$ such that we have the following property:

$$\nu \circ \phi_i = \tilde{\nu} \quad \text{in} \quad w_i. \tag{6.5.23}$$

Let $(\chi_i^\Gamma)_i$ be a $C^{m_\Gamma+1}$ unit partition of unity of the boundary Γ and for all i the set Γ_i is the interior of $\text{supp}(\chi_i^\Gamma)$. From this unit partition of unity , we can define a new one of the domain C_{0,η_0} defined for $x = x_\Gamma + \nu n(x_\Gamma)$ and i by $\chi_i(x) := \chi_i^\Gamma(x_\Gamma)$ and this unit partition is C^{m_Γ}.

We can chose the unit partition $(\chi_i)_i$ such that for all i there exists an open bounded set $w_i^\Gamma \subset \mathbb{R}^2$ and a $C^{m_\Gamma+1}$ diffeomorphism $\phi_i^\Gamma : w_i^\Gamma \subset \mathbb{R}^2 \mapsto \Gamma_i \subset \Gamma$. Then the interior of the support of the function χ_i is given by:

$$w_i := \Big\{x_\Gamma + \nu n(x_\Gamma),\ x_\Gamma \in \Gamma_i,\ \nu \in]0,\eta_0[\Big\},$$

and from this last set we can define the map $\phi_i w_i \mapsto w_i^\Gamma \times]0,\eta_0[$ for $x = x_\Gamma + \nu n(x_\Gamma) \in w_i$ by

$$\phi_i(x) := (\phi_i^\Gamma(x_i), \nu).$$

It is clear that this last function satisfies by construction the property (6.5.23).

Since $(\chi)_i$ is a unit partition of C_{0,η_0} we have:

$$\|u\|_{H^{s+2}(\omega)} \leq \sum_i \|\chi_i u\|_{H^{s+2}(\omega \cap w_i)},$$

and using that for all i the function χ_i is C^{s+2} and ϕ_i is a C^{s+2} diffeomorphism yields existence of C independent of u such that:

$$\|u\|_{H^{s+2}(\omega)} \leq C \sum_i \|(\chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')}, \tag{6.5.24}$$

where we defined for convenience the set $w_i' := \phi_i(\omega \cap w_i)$ Moreover since w_i' is an open subset of $\mathbb{R}^2 \times]0,\eta_0[$ we can apply (6.5.21) which leads that for all i

$$\|(\chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')} \leq C \operatorname{dist}(w_i', \mathbb{R}^2 \times \{0\})^{-(s+1)} \sup_{0 \leq s' \leq s} \|\nu^{s'+1} (\chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')}.$$

Combining this last estimate with the property (6.5.23) yields that for all i:

$$\|(\chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')} \leq C \operatorname{dist}(\omega \cap w_i, \Gamma)^{-(s+1)} \sup_{0 \leq s' \leq s} \|(\tilde{\nu}^{s'+1} \chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')},$$

and using that $\omega \cap w_i \subset \omega \Rightarrow \operatorname{dist}(\omega \cap w_i, \Gamma) \geq \operatorname{dist}(\omega, \Gamma)$ leads to:

$$\|(\chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')} \leq C \operatorname{dist}(\omega, \Gamma)^{-(s+1)} \sup_{0 \leq s' \leq s} \|(\tilde{\nu}^{s'+1} \chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')}.$$

Moreover reusing that ϕ_i is a C^{s+2} diffeomorphism leads existence of C independent of u such that for all i we have:

$$\|(\chi_i u) \circ \phi_i^{-1}\|_{H^{s+2}(w_i')} \leq C \operatorname{dist}(\omega, \Gamma)^{-(s+1)} \sup_{0 \leq s' \leq s} \|\tilde{\nu}^{s'+1} \chi_i u\|_{H^{s+2}(w_i)},$$

and combining with the regularity of χ_i leads that this last estimate become:

$$\|(\chi_i u)\circ\phi_i^{-1}\|_{H^{s+2}(w_i')} \le C\,\mathrm{dist}(\omega,\Gamma)^{-(s+1)} \sup_{0\le s'\le s} \|\tilde{\nu}^{s'+1} u\|_{H^{s+2}(w_i)}.$$

Thus combining this last estimate with (6.5.24) end the proof of estimate (6.5.21). □

6.6 Convergence of the method

Theorem 6.6.1. *If* (6.5.18) *is satisfied then the following estimate holds:*

$$\|u^\delta - u_h^\delta\circ\phi_h^\delta\|_{H^1(\Omega_\delta)} \le Ch.$$

Proof. Let $\Pi_h : C^0(\Omega_\delta^\delta) \mapsto V_h$ be the P1-interpolation operator and $\tilde{\Pi}_h : C^0(\Omega_\delta) \mapsto \tilde{V}_h := \{X\circ\phi_h,\ X\in V_h\}$ defined for $u\in C^0(\Omega_\delta)$ by:

$$\tilde{\Pi}_h X := \Pi_h\left(X\circ\phi_h^{-1}\right)\circ\phi_h.$$

First let us prove existence of $C>0$ such that for all $h,\delta>0$ the following estimate holds:

$$\|u^\delta - \tilde{\Pi}_{T_h} u^\delta\| \le C\cdot h. \tag{6.6.25}$$

Thanks to [28, Theorem 15.3] for all $T\in T_h$ the following estimate holds:

$$\left\|u^\delta - \tilde{\Pi}_{T_h} u^\delta\right\|_{H^1(T)} \le Ch_T\left\|u^\delta\right\|_{H^2(T)} \tag{6.6.26}$$

Thus thanks to Proposition 6.4.1 we get existence of $C>0$ such that for all $\delta,h>0$ we have:

$$\mathrm{dist}(T,\Gamma)\le\delta \Rightarrow \left\|u^\delta - \tilde{\Pi}_{T_h} u^\delta\right\|_{H^1(T)} \le C\left\|\tilde{\nu}\cdot u^\delta\right\|_{H^2(T)}\cdot h. \tag{6.6.27}$$

Moreover thanks to Proposition 6.5.1 we get existence of $C>0$ such that for all $h,\delta>0$ we have:

$$\forall T\in T_h,\ \left\|u^\delta\right\|_{H^2(T)} \le C\,\mathrm{dist}(T,\Gamma)^{-1}\cdot\left\|\tilde{\nu}\cdot u^\delta\right\|_{H^2(T)},$$

which leads combined with (6.6.25) and the constraint (6.5.18) to:

$$\forall T\in T_h,\ \left\|u^\delta - \tilde{\Pi}_{T_h} u^\delta\right\|_{H^1(T)} \le C\frac{(\delta\,\mathrm{dist}(T,\Gamma)^{-1}+1)}{2}\left\|\tilde{\nu}\cdot u^\delta\right\|_{H^2(T)}\cdot h.$$

Therefore we get the following implication:

$$\forall T\in T_h,\ \mathrm{dist}(T,\Gamma)\ge\delta \Rightarrow \left\|u^\delta - \tilde{\Pi}_{T_h} u^\delta\right\|_{H^1(T)} \le C\left\|\tilde{\nu}\cdot u^\delta\right\|_{H^2(T)}\cdot h,$$

and combining this implication with (6.6.27) end the proof of the estimate (6.6.25).

Using similar argument from the proof of Proposition 5.4.3 yields existence of $h_0>0$ and $\eta_0>0$ independent of δ and h such that for all $x_h^\delta\in V_h^\delta$ we have:

$$|h|<h_0 \quad\Rightarrow\quad \|x_h^\delta\|_{H^1(\Omega_\delta^h)} \le \eta_0 \sup_{\substack{y_h\in V_h^\delta\\ \|y_h^\delta\|_{H^1(\Omega_\delta^h)}=1}} a_h^\delta(x_h^\delta, y_h^\delta),$$

which leads to:

$$\left\|u^\delta \circ (\phi_h^\delta)^{-1} - u_h^\delta\right\|_{H^1(\Omega_\delta^h)} \leq \left\|u^\delta - \tilde{\Pi}_{T_h} u^\delta\right\|_{H^1(\Omega_\delta)} + \eta_0 \sup_{\substack{y_h \in V_h^\delta \\ \|y_h^\delta\|_{H^1(\Omega_\delta^h)}=1}} a_h^\delta\left(\tilde{\Pi}_{T_h} u^\delta \circ (\phi_h)^{-1} - u_h^\delta, y_h^\delta\right).$$

Combining this last estimate with (6.6.25) and the uniform continuity of the sesquilinear form q_h^δ yields:

$$\left\|u^\delta \circ (\phi_h^\delta)^{-1} - u_h^\delta\right\|_{H^1(\Omega_\delta^h)} \leq Ch + \eta_0 \sup_{\substack{y_h \in V_h^\delta \\ \|y_h^\delta\|_{H^1(\Omega_\delta^h)}=1}} a_h^\delta\left(u^\delta \circ (\phi_h)^{-1} - u_h^\delta, y_h^\delta\right),$$

which leads combined with (6.2.2) to:

$$\begin{aligned}\left\|u^\delta \circ (\phi_h^\delta)^{-1} - u_h^\delta\right\|_{H^1(\Omega_\delta^h)} &\leq Ch + \eta_0 \sup_{\substack{y_h \in V_h^\delta \\ \|y_h^\delta\|_{H^1(\Omega_\delta^h)}=1}} a_h^\delta\left(u^\delta \circ (\phi_h)^{-1}, y_h^\delta\right) - (f, y_h^\delta \circ \phi_h^\delta), \\ &\leq Ch + \eta_0 \sup_{\substack{y_h \in V_h^\delta \\ \|y_h^\delta\|_{H^1(\Omega_\delta^h)}=1}} a_h^\delta\left(u^\delta \circ (\phi_h)^{-1}, y_h^\delta\right) - a^\delta(u^\delta, y_h^\delta \circ \phi_h^\delta).\end{aligned}$$

Therefore it remains to prove that the following estimate holds:

$$\sup_{\substack{y_h \in V_h^\delta \\ \|y_h^\delta\|_{H^1(\Omega_\delta^h)}=1}} a_h^\delta\left(u^\delta \circ (\phi_h)^{-1}, y_h^\delta\right) - a^\delta(u^\delta, y_h^\delta \circ \phi_h^\delta) \leq Ch. \tag{6.6.28}$$

To simplify the writing we assume that $k = 0$ because the generalization for $k \neq 0$ is obvious. For all $y_h \in V_h^\delta$ we have that the quantity: $E(y_h^\delta) := a_h^\delta\left(u^\delta \circ (\phi_h)^{-1}, y_h^\delta\right) - a^\delta(u^\delta, y_h^\delta \circ \phi_h^\delta)$

$$\begin{aligned}E(y_h^\delta) &= \int_{\Omega_\delta^h} \rho_h^\delta \nabla\big(u^\delta \circ (\phi_h)^{-1}\big) \cdot \nabla y_h^\delta d\Omega_\delta^h - \int_{\Omega_\delta} \rho^\delta \nabla\big(u^\delta\big) \cdot \nabla\big(y_h^\delta \circ \phi_h\big) d\Omega_\delta, \\ &= \int_{\Omega_\delta} \Big(D_\Omega \nabla u^\delta, \nabla\big(y_h^\delta \circ \phi_h\big)\Big) d\Omega_\delta\end{aligned}$$

where we defined the tensor field $D_\Omega := \rho_h^\delta \circ \phi_h \cdot \mathrm{D}\,\phi_h^{-1} \cdot \mathrm{D}\,\phi_h^{-\dagger} |\det(\mathrm{D}\,\phi_h)| - \rho^\delta$. Therefore it is sufficient to prove that:

$$\|D_\Omega\|_{L^\infty(\Omega_\delta)} \leq Ch. \tag{6.6.29}$$

Indeed from Proposition 5.4.1 we get that

$$\left\|\mathrm{D}\,\phi_h^{-1}\,\mathrm{D}\,\phi_h^{-\dagger} |\det(\mathrm{D}\,\phi_h)| - 1\right\|_{L^\infty(\Omega_\delta)} \leq Ch. \tag{6.6.30}$$

Moreover to (6.5.18) and the regularity at least $C^1(\Gamma; [0,1] \times [-,0])$ of the reference function $\hat{\rho}$ we have:

$$\|\rho_h^\delta \circ \phi_h - \rho^\delta\|_{L^\infty(\Omega_\delta)} \leq C\|\rho^\delta\|_{W^\infty(C_\delta)} \sup_{T \in T_h^\delta, T \subset \Omega_\delta^h \setminus \Omega_h} h_T \leq C\delta^{-1} \sup_{T \in T_h^\delta, T \subset \Omega_\delta^h \setminus \Omega_h} h_T \leq Ch.$$

Thus combining this last estimate with (6.6.30) end the proof of (6.6.29) which conclude whole the proof. □

Chapter 7

Numerical validation of the approximate boundary conditions

Here we present numerical comparison between the exact model and the approximate model.

7.1 Algorithm for construction of mesh and computation of local coordinate

We emphasize difficulties to mesh our geometry and computing our coefficient. Indeed the thin coat and the function ρ^δ and μ^δ are defined from the local coordinate function $\mathcal{L}$. Indeed we recall that:

$$C_\delta = \mathcal{L}^{-1}(\Gamma\times]-\delta,0[),$$

and for all $x \in C_\delta$:

$$\rho^\delta(x) = \hat{\rho}\left(x_\Gamma, \frac{\psi_\Gamma(x_\Gamma)}{\delta}, \frac{\hat{\nu}}{\delta}\right),$$

with $(x_\Gamma, \nu) = \mathcal{L}(x)$. We recall that this last function is implicitly defined by its inverse $\mathcal{L}^{-1}$ given by:

$$\mathcal{L}^{-1}(x_\Gamma, \nu) = x_\Gamma + \nu n(x_\Gamma).$$

Although the quantity x_Γ can be explicitly defined by the minimizer of the functional defined on Γ by

$$x_\Gamma \mapsto |x - x_\Gamma|,$$

and ν be defined by:

$$\nu = (x - x_\Gamma, \nu),$$

it is easy to see that a brute-force using this last formula will be very slow. Indeed let N be the number of point of discretization of C_δ. Then for $x \in C_\delta$, an exhaustive research of the closest point $x_\Gamma \in \Gamma$ has a average times proportional to N. Since the method will do it for each point of the discretization we get that the complexity of this brute-force method is $O(N^2)$. Moreover if we assume that δ is very small, therefore N will must very large compared to $1/\delta$. We will see that we typically take $N = 10000 \times 20$ and then the computer will do 40 000 000 000 operations.

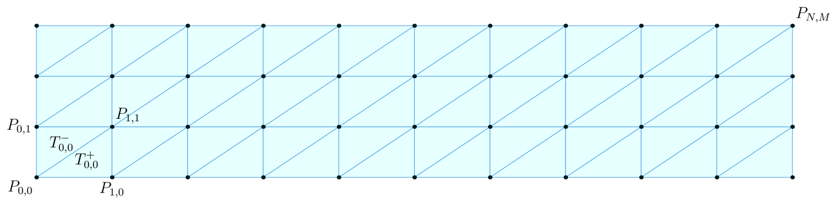


Figure 7.1: The rectangular triangulation

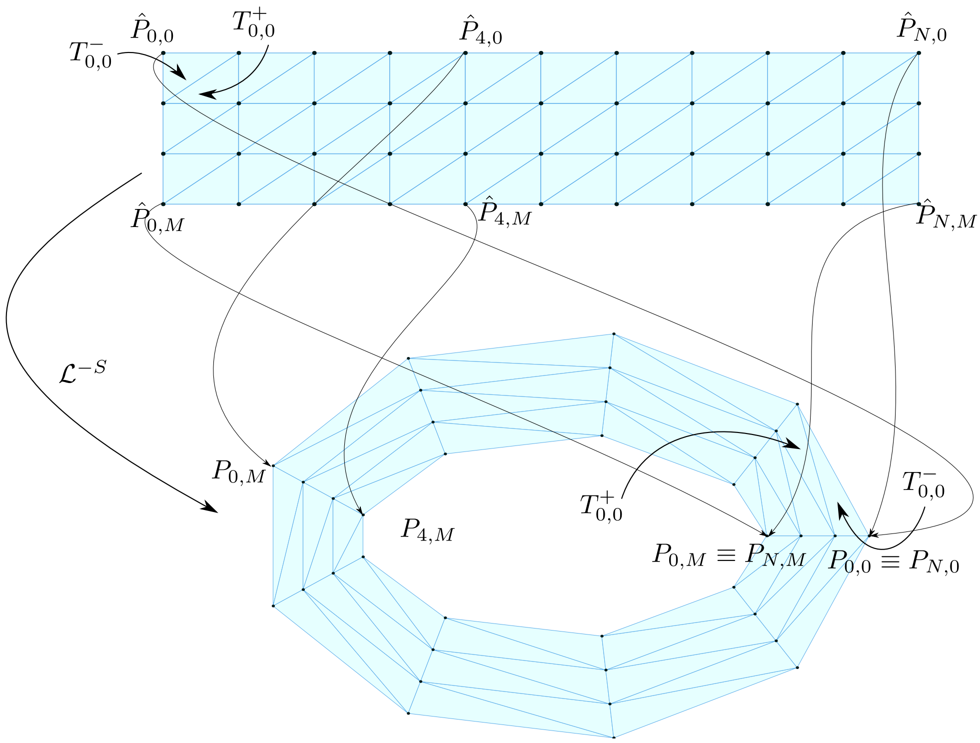


Figure 7.2: The transformation of mesh

Now this last difficulty is described, let us explain the way we use to overcome it. First define for δ, N, M $\hat{T}_{\delta,N,M}$ the rectangular triangulation by the mesh of $]0,1[\times]-\delta,0[$ whose vertex is the collection $(P_{ij})_{0\leq i\leq N, 0\leq j\leq M}$ defined for $0 \leq i \leq N, 0 \leq j \leq M$ by:

$$\hat{P}_{ij} := (i/N, -j\delta/M),$$

and the triangles which are the collection $(\hat{T}^+_{ij}, \hat{T}^-_{ij})_{0\leq i\leq N-1, 0\leq j\leq M-1}$ defined for $0 \leq i \leq N-1, 0 \leq j \leq M-1$ by (see Figure 7.1):

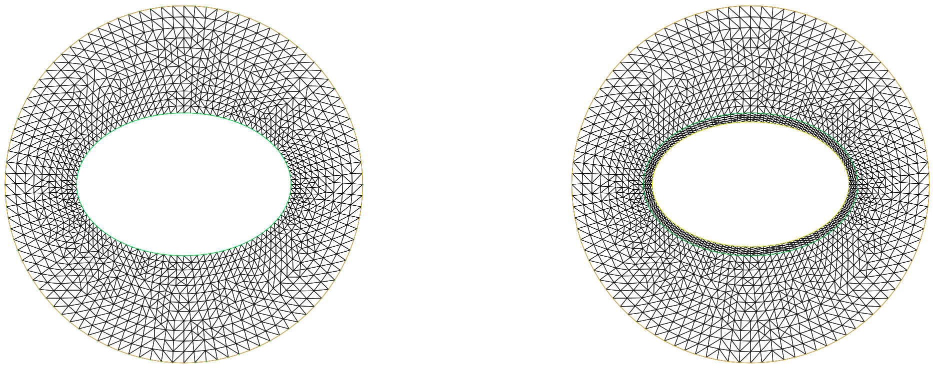
$$\hat{T}^+_{ij} := \Big\{(i,j),(i+1,j),(i,j+1)\Big\} \quad \text{and} \quad \hat{T}^-_{ij} := \Big\{(i+1,j+1),(i+1,j),(i,j+1)\Big\}.$$

Thus from this last mesh, we construct a new mesh $T_{\delta,N,M}$ defined by the transformation by the function $\mathcal{L}^{-\mathcal{S}} : [0,1[\times]-\delta,0[\mapsto C_\delta$ defined for $(t,\nu) \in [0,1[\times]-\delta,0[$ by :

$$\mathcal{L}^{-\mathcal{S}}(t,\nu) := S(t) - \nu \cdot n(S(t)),$$

of the mesh $\hat{T}_{\delta,N,M}$. More precisely the vertex of this mesh is defined by the mesh whose vertex is the following collection:

$$(P_{ij})_{ij} := (\mathcal{L}^{-\mathcal{S}}(\hat{P}_{ij}))_{ij},$$



(a) The mesh T_h (b) The mesh T_h^δ

Figure 7.3: Example of meshing of the thin coat

and the triangles are the same as the initial mesh $\hat{T}_{\delta,N,M}$ (see Figure 7.2).

We implemented a little program written in `C++` that take an initial mesh T_h of the domain Ω whose format is "`.msh`" and builds a new mesh T_h^δ of the domain Ω_δ with same format. We refer the reader to Figure 7.3 for an example of application of this last algorithm.

The initial mesh T_h is represented by a triplet (V_h, T_h, Γ_h) where:

- N_h^v and N_h^t are the number of vertex and triangle of the mesh T_h.

- $V_h \in (\mathbb{R}^2)^{N_h^v}$ is a finite sequence of point of $\mathbb{R}^2$ containing the vertex of the mesh T_h.

- The triangles are represented by element of $\mathcal{P}_3$ which the set of subset of $(\{1, \cdot, N_h^v\})$ whose cardinal is 3. All $\{i_1, i_2, i_3\} \in \mathcal{P}_3$ represent the triangle whose corners are $V_h(i_1), V_h(i_2)$ and $V_h(i_3)$.

- $T_h \in \left(\mathcal{P}_3(\{1, \cdot, N_h^v\})\right)^{N_h^t}$ is a finite sequence of $\mathcal{P}_3(\{1, \cdot, N_h^v\})$ containing the triangle of the mesh T_h.

- N_h^Γ is the number of point of discretization of the boundary Γ.

- $\Gamma_h \in \{1, \cdot, N_h^v\}^{N_h^\Gamma}$ is a finite sequence such that for all $i \in \{1, \cdot, N_h^v\}$ the point $V_h(i)$ belongs to Γ.

For any set E and natural n, the concatenation operator $\cdot$ is defined for all $(L_1, \cdots, L_n) \in E^n$ and $L_{n+1} \in E$ by $(L_1, \cdots, L_n) \cdot L_{n+1} := (L_1, \cdots, L_n, L_{n+1})$. We introduced this last operator in order to represent the insertion operation. Therefore we can give the algorithm we use to construct the new mesh $(V_h^\delta, T_h^\delta, \Gamma_h^\delta)$:

Data: (V_h, T_h, Γ_h)
Result: $(V_h^\delta, T_h^\delta, X_{\Gamma,h}, \nu_h)$
$(V_h^\delta, T_h^\delta) \leftarrow (V_h, T_h)$; `Initialization with copying the initial mesh`;
for $1 \le i \le N_h^v$ **do**
    $P \leftarrow V_h(\Gamma_h(i))$;
    $N \leftarrow \mathrm{normal}(V_h(\Gamma_h(i-1)), V_h(\Gamma_h(i+1))$; `Approximation of the unit normal` ;
    for $1 \le j \le N_c$ **do**
        $\nu_j \leftarrow -\delta \cdot j/N_c$;
        $P_\nu \leftarrow P + \nu_j \cdot N$;
        $V_h^\delta \leftarrow V_h^\delta \cdot P_\nu$; `We push the the vertex on the mesh` T_h^δ;
        $(I_1, I_2, I_3, I_4) \leftarrow \mathrm{Carre}(V_h, i, j)$;
        $t_h^1 \leftarrow \{I_1, I_2, I_3\}$;
        $t_h^2 \leftarrow \{I_2, I_3, I_4\}$;
        $T_h \leftarrow T_h \cdot t_h^1$;
        $T_h \leftarrow T_h \cdot t_h^2$;
        $X_{\Gamma,h}(I_1) \leftarrow i/N_h$;
        $\nu_h(I_1) \leftarrow \nu$;
    end
end

Data: (P_{-1}, P_1)
Result: N
$DP \leftarrow V_h(\Gamma_h(i+1)) - V_h(\Gamma_h(i-1))$;
$DP \leftarrow DP/|DP|$;
$N \leftarrow (-Dp_2, Dp_1)$

Data: (P_{-1}, P_1)
Result: N
if $j = 1$ **then**
    $I_1 \leftarrow V_h(\Gamma_h(i)$;
    $I_2 \leftarrow V_h(\Gamma_h(i+1)$;
else
    $I_1 \leftarrow N_h^v + (i-1) \cdot N_c + j - 1$;
    $I_2 \leftarrow N_h^v + i \cdot N_c + j - 1$;
end
$I_3 \leftarrow N_h^v + (i-1) \cdot N_c + j$;
$I_4 \leftarrow N_h^v + i \cdot N_c + j$;

7.2 Configuration

- The reference function function are defined by $\mu = 1$ and for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times [0,1] \times [-1, 0]$ by:
$$\hat{\rho}(x_\Gamma; \hat{x}, \hat{\nu}) := 1.25 + \cos(2\pi\hat{x}).$$
- The function g is defined for $x := \big(\cos(\theta), \sin(\theta)\big) \in \partial\mathbb{B}$ by $g(x) := \cos(5\theta)$.

- The kite is the region in the case where the application $P : [0,1] \mapsto \mathbb{R}^2$ is defined for $t \in [0,1]$ by:
$$P(t) := \big(\cos(t) + 0.3 \cdot \cos(2 \cdot t), \sin(t)\big)$$
- The ellipse is the form in the case where the application $P : [0,1] \mapsto \mathbb{R}^2$ is defined for $t \in [0,1]$ by:
$$P(t) := \big(0.6 \cdot \cos(t), 0.4 \cdot \sin(t)\big)$$
- For meshing the domain Ω_δ we always take 1500 point for the discretization of $\partial\mathbb{B}$.
- For meshing the domains Ω_δ we always take 60000×30 of discretization of the thin coat C_δ

7.3 Graphical comparison for thin mesh of the approximated solutions

Here we take to approximate the domain Ω, 10000 points of discretization on the boundary Γ. We present numerical results in Figure 7.4 for $k = 10$ in the case of a kite. In this case we don't see difference between the exact solution, the first order solution and the second order solution for two values of δ. For the two values of δ figures are very similar but present small differences. Finally we do not see a difference in the case of the ellipse presented in Figure 7.6.

7.4 Graphical plot of the error:

Let $\eta > 0$ and $K_\eta := \{x \in \Omega,\ \mathrm{dist}(x, \Gamma) > \eta\}$. We introduce a regular third mesh T_h^{err} of the domain K_η. Let $\Pi_\eta^{\mathrm{err}} : H^1(\mathbb{R}^2) \mapsto V_{T_h^{\mathrm{err}}}$ be an projection application the space $V_{T_h^{\mathrm{err}}}$ of P_1 function on this last mesh. The error is defined for $\eta > 0$ and $i = 1, 2$ by:

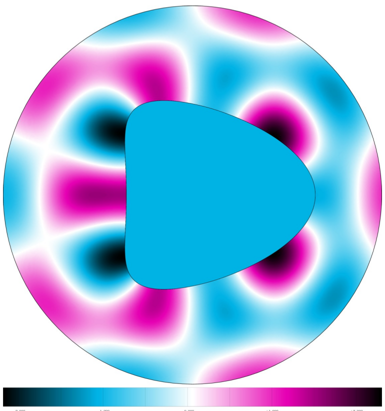
$$\epsilon_\eta^i := \Pi_\eta^{\mathrm{err}}(u_h^\delta - u_{i,\boldsymbol{h}}^\delta)$$

7.5 Numerical rate of convergence

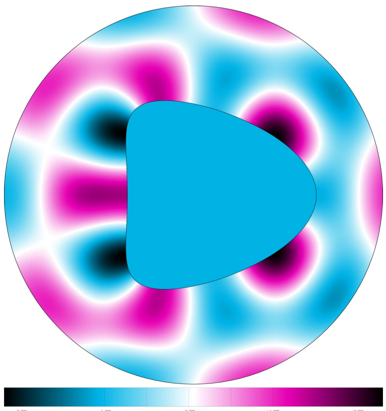
We seek to numerically show the estimate (4.5.91). To do that emphasize this last estimate is equivalent to the existence of C such that we have for all $i = 1, 2$ and $\delta > 0$ the following estimate:

$$\ln\big(\|u^\delta - u_i^\delta\|_{H^1(K)}\big) \leq (i+1) \cdot \ln(\delta) + C.$$

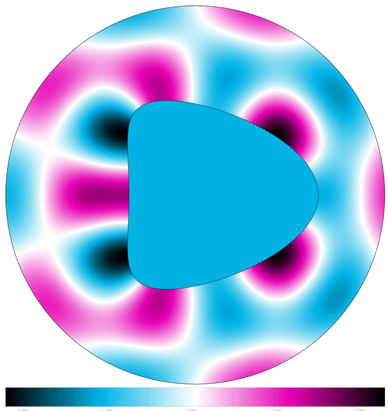
Thus we produce for the same geometry and right hand-side several simulation for several small $(\delta_j)_j$ and we do a linear regression between vector $(\ln(\delta_i))_i$ and $\Big(\ln\big(\|u^{\delta_j} - u_i^{\delta_j}\|_{H^1(K)}\big)\Big)_j$ and we compare the slope with the theoretical slope $i+1$. From Figure 7.7 the slope comparison for a kite looks good but looks worst for an ellipse.



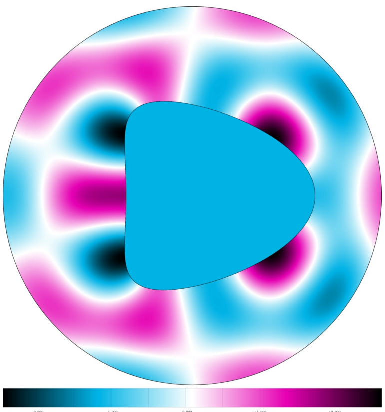
(a) exact solution, $\delta = 0.01$



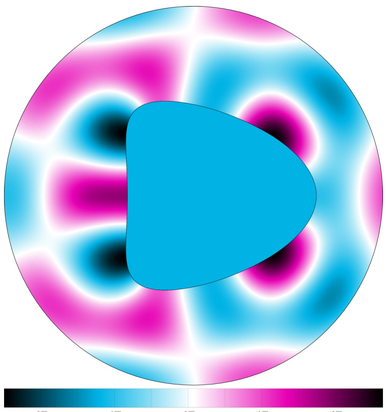
(b) order 1, $\delta = 0.01$



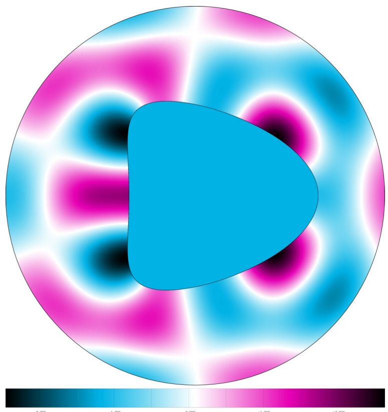
(c) order 2, $\delta = 0.01$



(d) exact solution $\delta = 0.02$

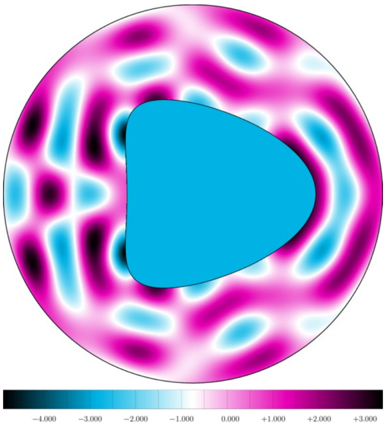


(e) order 1, $\delta = 0.02$

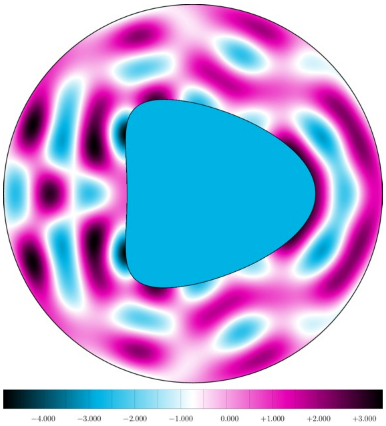


(f) order 2, $\delta = 0.02$

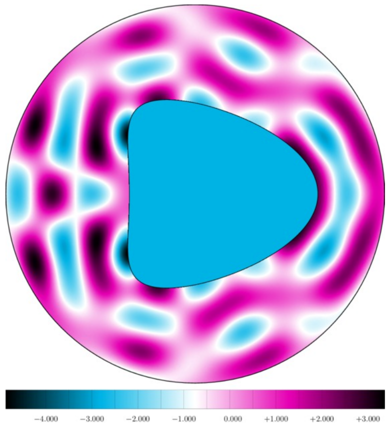
Figure 7.4: Graphical comparison for the kite and $k = 10$



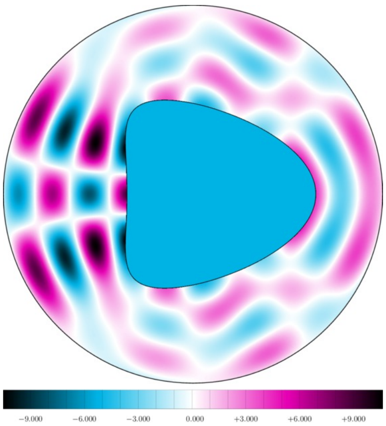
(a) exact solution, $\delta = 0.01$



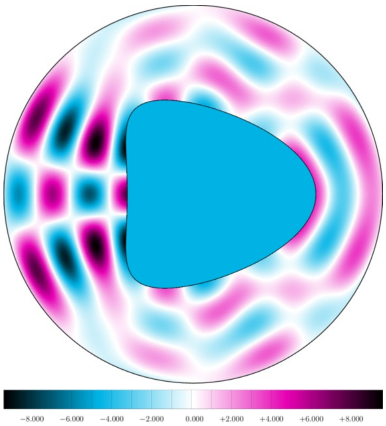
(b) order 1, $\delta = 0.01$



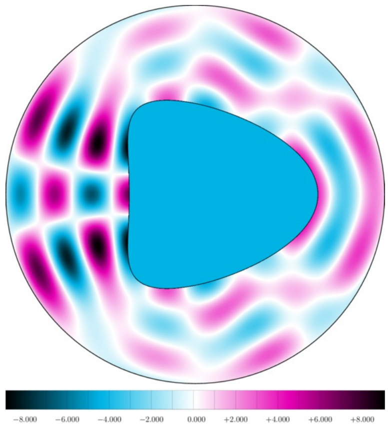
(c) order 2, $\delta = 0.01$



(d) exact solution $\delta = 0.02$

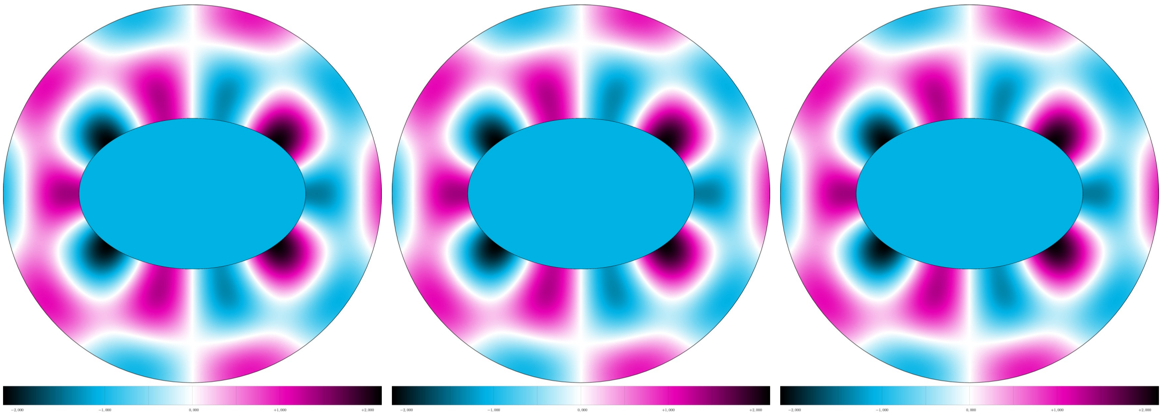


(e) order 1, $\delta = 0.02$

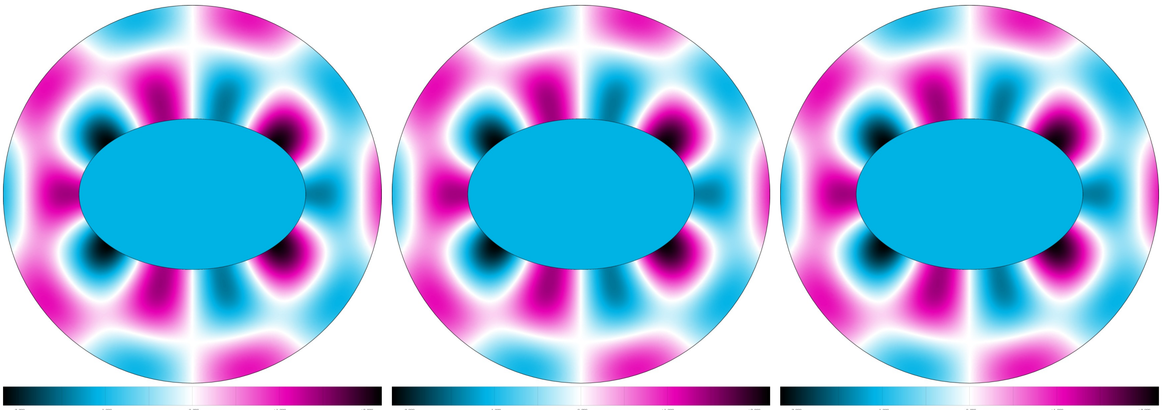


(f) order 2, $\delta = 0.02$

Figure 7.5: Graphical comparison for the kite and $k = 20$

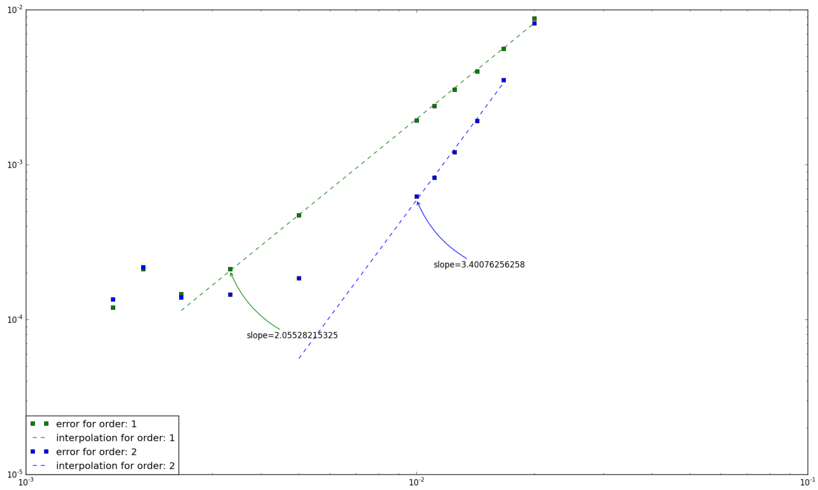


(a) exact solution, $\delta = 0.01$ (b) order 1, $\delta = 0.01$ (c) order 2, $\delta = 0.01$

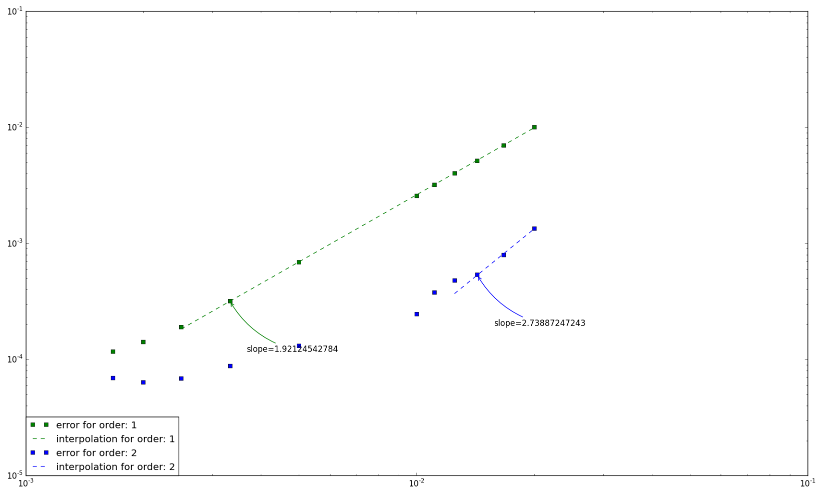


(d) exact solution $\delta = 0.02$ (e) order 1, $\delta = 0.02$ (f) order 2, $\delta = 0.02$

Figure 7.6: Graphical comparison for the ellipse and $k = 10$



(a) The kite



(b) The ellipse

Figure 7.7: Numerical rate convergence

Part III

The case of 3D Maxwell equations

Table of Contents

Chapter 8

Description of the Maxwell's equations and objective

We have extended part I and II for the scalar case to the vectorial case.

8.1 Description of the studied problem

Let us start with a quick description of the geometry of our problem and a presentation of the model problem. Let O be a bounded domain of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus O$ is connected with C^∞ boundary Γ and $\delta > 0$. We call the "thin coating of width δ" the following subset C^δ of O:

$$C^\delta := \{x \in O,\ \mathrm{dist}(x, \Gamma) < \delta\}.$$

Here the quantity $\mathrm{dist}(x, \Gamma)$ is the distance of x from the surface Γ defined by

$$\mathrm{dist}(x, \Gamma) := \inf_{x_\Gamma \in \Gamma} |x - x_\Gamma|,$$

and $|.|$ is the classical euclidean norm of $\mathbb{R}^3$. We need to introduce the complement of O in $\mathbb{R}^3$ $\Omega := \mathbb{R}^3 \setminus \overline{O}$ and $\Omega^\delta := \overline{\Omega} \cup C^\delta$. We refer the reader to the Figure 8.1 for an illustration in 2D. The electric and magnetic fields are respectively denoted by E^δ and H^δ. We introduce the curl operator defined for vector field:

$$E : (x_1, x_2, x_2) \mapsto \begin{pmatrix} E_1(x_1, x_2, x_2) \\ E_2(x_1, x_2, x_2) \\ E_3(x_1, x_2, x_2) \end{pmatrix},$$

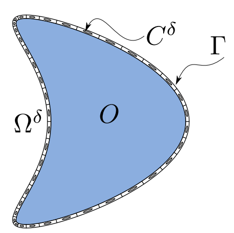


Figure 8.1: Illustration of the geometry

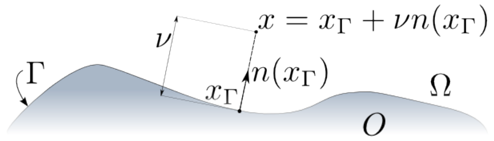


Figure 8.2: Illustration of (x_Γ, ν)

by:

$$\mathbf{rot}E := \begin{pmatrix} \partial_{x_3}E_2 - \partial_{x_2}E_3 \\ \partial_{x_1}E_3 - \partial_{x_3}E_1 \\ \partial_{x_2}E_1 - \partial_{x_1}E_2 \end{pmatrix}.$$

The problem that we are interested in is the following: Find $(E^\delta, H^\delta) \in H_{\text{loc}}(\mathbf{rot})^2$ such that:

$$\mathbf{rot}E^\delta = -ik\mu^\delta H^\delta \quad \text{and} \quad \mathbf{rot}H^\delta = ik\epsilon^\delta E^\delta + J_{\text{source}} \text{ in } \Omega^\delta, \tag{8.1.1}$$

with the following boundary conditions:

$$E^\delta \times n^\delta = 0 \quad \text{and} \quad \mu^\delta H^\delta \cdot n^\delta = 0 \quad \text{on} \quad \Gamma_\delta := \partial\Omega^\delta, \tag{8.1.2}$$

and E^δ, H^δ satisfy the Silver-Muller radiation condition (See [59] and [69]):

$$\lim_{|x| \mapsto \infty} |x| \left(H^\delta(x) \times \frac{x}{|x|} - E^\delta(x) \right) = 0. \tag{8.1.3}$$

Here n^δ and n are respectively the unit outward normal to $\partial\Omega^\delta$ and Ω, $k \in \mathbb{R}$ is the wave-number and J_{source} denotes a given current source. Moreover ϵ^δ, μ^δ denote the characteristics of the medium supposed to be equal to 1 in Ω. These function are supposed to be $\psi_\Gamma - \delta -$ periodic in the thin coating C^δ associated to reference function $\hat{\epsilon}$ and $\hat{\mu}$. We recall that it means that for all $\in C^\delta$ we have

$$\epsilon^\delta(x) = \hat{\epsilon}(x_\Gamma; \hat{x}, \hat{\nu}) \quad \text{and} \quad \mu^\delta(x) = \hat{\mu}(x_\Gamma; \hat{x}, \hat{\nu})$$

where $(x_\Gamma, \nu) \in \Gamma \times] - \delta, 0[$ are the unique solution of $x = x_\Gamma + \nu n(x_\Gamma)$ and $(\hat{x}, \hat{\nu}) := \dfrac{\big(\psi_\Gamma(x_\Gamma), \nu\big)}{\delta}$. (see Figure 8.2). In this work the map $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$ is a data of our problem. We recall that (8.1.2) means that we use a model of perfect conductor and (8.1.3) mean that the scattered waves is outgoing.

8.2 Assumption on the coefficient and the map ψ_Γ

We assume for technical reason that:

$$\inf \hat{\epsilon} > 0 \quad \text{and} \quad \inf \hat{\mu} > 0, \tag{8.2.4}$$

to ensure that our problem is well posed. (See [58]). We will see that the following regularity well simplify our analysis:

$$(\hat{\epsilon}, \hat{\mu}) \in C^\infty\left(\Gamma; L^\infty(\hat{Y}_\infty)\right)^2 \quad \text{and} \quad \psi \in C^\infty(\Gamma)^2, \tag{8.2.5}$$

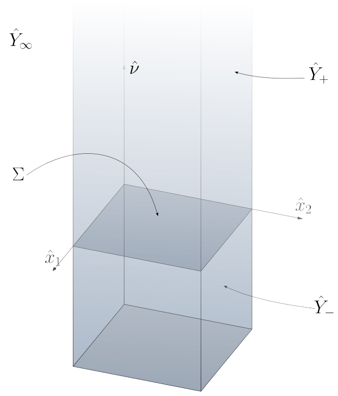


Figure 8.3: Illustration the of infinite strip

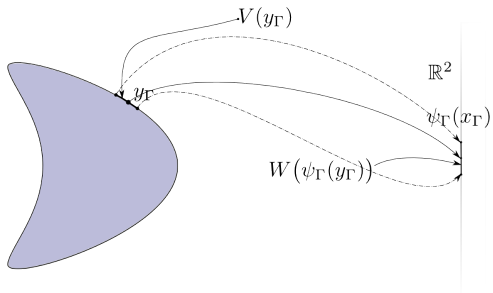


Figure 8.4: Illustration of the map $\psi_\Gamma : \Gamma \mapsto \mathbb{R}^2$

where we introduced the infinite strip $\hat{Y}_\infty :=]0,1[^2\times]-,\infty[.$ (see Figure 8.3). We assume the existence of a open set $\Gamma_\mathrm{M} \subset \Gamma$ such that for all $y_\Gamma \in \overline{\Gamma_\mathrm{M}}$ there exist neighborhoods $V(y_\Gamma) \subset \Gamma$, $W\big(\psi_\Gamma(x_\Gamma)\big) \subset \mathbb{R}^2$ of y_Γ and 0 such that $\psi_\Gamma : V(y_\Gamma) \mapsto W\big(\psi_\Gamma(y_\Gamma)\big)$ is a C^∞ diffeomorphism.(See Figure 8.4). Our coefficient $\hat{\epsilon}$ and $\hat{\mu}$ are supposed "patching admissible". We recall that it mean that for all $x_\Gamma \notin \Gamma$ the map $(\hat{x}, \hat{\nu}) \mapsto \hat{\epsilon}(x_\Gamma; \hat{x}, \hat{\nu})$ does not depend of $\hat{x}$.

Finally we impose that $\mathrm{supp}(J_\mathrm{source}) \cap \Gamma = \emptyset$.

8.3 Effective boundary conditions

The objective of this work is to find an operator $\mathcal{Z}$ defined on some space of functions defined on Γ and takes values in some space of functions defined Γ such that if we delete from our geometry the thin coat C_δ and we replace by what we call the impedance boundary condition:

$$\gamma_t E^\delta = ik\mathcal{Z}\big(\gamma_T H^\delta\big),$$

where we define for function $u : \Omega \mapsto \mathbb{C}^3$ the two traces for $x_\Gamma \in \Gamma$ by:

$$\gamma_T u(x_\Gamma) := \big(n(x_\Gamma) \times u(x_\Gamma)\big) \times n(x_\Gamma) \quad \text{and} \quad \gamma_t u(x_\Gamma) := u(x_\Gamma) \times n(x_\Gamma),$$

then the new scattered fields (E^δ, H^δ) are a good approximation of the exact field.

The case of uniform coefficient in the thin coat has already been studied in [44] and [14].

8.4 Summary of the work

The main steps of our work are as follows:

1. By using the method of matched asymptotic expansions, we construct a sequence of vector fields $(E^n, H^n)_{n\in\mathbb{N}}$ such that we have for all $n \in \mathbb{N}$ the following estimate:

$$E^\delta - \sum_{k=0}^{n} \delta^k E^k = O(\delta^{n+1}) \quad \text{and} \quad H^\delta - \sum_{k=0}^{n} \delta^k H^k = O(\delta^{n+1}), \tag{8.4.6}$$

2. We identify an operator $\mathcal{Z}_1$ such that we have for all $0 \leq n' \leq n$

$$\gamma_t E^1 = ik\mathcal{Z}_1 \gamma_T H^0 \quad \text{and} \quad \gamma_t E^0 = 0 \text{ on } \Gamma.$$

3. We emphasize that this last equality yields:

$$\gamma_t \left(E^0 + \delta E^1\right) - ik\mathcal{Z}_n^\delta \left(\gamma_T(H^0 + \delta H^1)\right) = O(\delta^2). \tag{8.4.7}$$

4. We prove that the approximated boundary condition is stable. This means that there exists $C > 0$ independent of δ such that for all $E, H : \Omega \mapsto \mathbb{C}^3$ satisfying the Maxwell equation (8.1.1) with $J_{\text{source}} = 0$ and the radiating condition (8.1.3), we have estimates of the form:

$$\|E\| + \|H\| \leq C\|g_\Gamma\| \quad \text{with} \quad g_\Gamma := \gamma_t E - \delta ik\mathcal{Z}_1 \left(\gamma_T H\right). \tag{8.4.8}$$

5. We introduce the function $(E_1^\delta, H_1^\delta) : \Omega \mapsto \mathbb{C}^3$ defined as the unique solution of the Maxwell equation (8.1.1) and the radiating condition (8.1.3) such that we have on Γ:

$$\gamma_t E_1^\delta - \delta ik\mathcal{Z}_1 \gamma_T H_1^\delta = 0. \tag{8.4.9}$$

6. We combine (8.4.8) with (8.4.9) which leads to

$$\gamma_t \left(E^0 + \delta E^0 - E_1^\delta\right) - ik\delta\mathcal{Z}_1 \left(\gamma_T(E^0 + \delta E^0 - E_1^\delta)\right) = O(\delta^2). \tag{8.4.10}$$

   Combining this with (8.4.7) yields:

$$\left\|E^0 + \delta E^1 - E_1^\delta\right\| + \left\|H^0 + \delta H^1 - E_1^\delta\right\| = O(\delta^2).$$

7. We combine this last estimate with (8.4.6) which yields by using the triangle inequality the final estimate:

$$E^\delta - E_1^\delta = O(\delta^2) \quad \text{and} \quad H^\delta - H_1^\delta = O(\delta^{n+1}).$$

Chapter 9

Formal asymptotic expansion

9.1 Summary of the matching expansion method in the Maxwell case

This method consists in seeking two asymptotic expansions of the solution. One is valid near the boundary called the near-field expansion and the other is valid far from the boundary called far-field expansion. Firstly let us chose a function $\eta : \delta \mapsto \eta(\delta)$ such that:

$$\lim_{\delta \to \infty} \eta(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to \infty} \frac{\eta(\delta)}{\delta} = \infty, \tag{9.1.1}$$

and define the following zones (see Figure 9.1 for a graphical illustration of these regions):

- The far-field zone is defined by $\{x \in \Omega, \operatorname{dist}(x, \Gamma) > 2\eta\}$. For all point x in this zone we assume that our field take the following form:

$$E^\delta(x) = \sum_{n \in \mathbb{N}} \delta^n E^n(x) \quad \text{and} \quad H^\delta(x) = \sum_{n \in \mathbb{N}} \delta^n H^n(x), \tag{9.1.2}$$

  where for all $n \in \mathbb{N}$ the functions E^n and H^n are defined on Ω.

- The near-field zone is defined by $C_\delta \cup \{x \in \overline{\Omega}, \operatorname{dist}(x, \Gamma) < \eta_0\}$. In this zone, we have to take into account the $\psi_\Gamma - \delta-$periodicity of our physical coefficient $\mu^\delta, \epsilon^\delta$. That is why we use a more complicated ansatz inspired from the periodic homogenization [5].

  In this zone we formally assume that E^δ and H^δ are a series of $\psi_\Gamma - \delta-$periodic function. That means that for all point x in this zone we have:

$$E^\delta(x) = \sum_{n=0}^{\infty} \delta^n \underline{\hat{E}}^n(x_\Gamma; \hat{x}, \hat{\nu}) \quad \text{and} \quad H^\delta(x) = \sum_{n=0}^{\infty} \delta^n \underline{\hat{H}}^n(x_\Gamma; \hat{x}, \hat{\nu}), \tag{9.1.3}$$

  where (x_Γ, ν) are the unique solution of $x = x_\Gamma + \nu n(x_\Gamma)$ (see Figure 8.2) and

$$(\hat{x}, \hat{\nu}) := \frac{\big(\psi_\Gamma(x_\Gamma), \nu\big)}{\delta}. \tag{9.1.4}$$

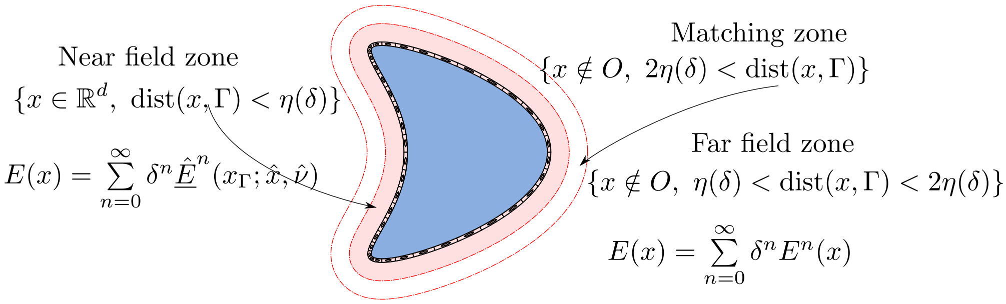


Figure 9.1: Illustration of the three zones

Here for all n in $\mathbb{N}$ the function $\underline{\hat{E}}^n, \underline{\hat{H}}^n : \Gamma \mapsto P(\hat{\Omega})$ is defined from Γ into $P(\hat{\Omega})$ and we recall that $P(\hat{\Omega})$ is the set of function defined on $\hat{\Omega} := \mathbb{R}^2 \times]-1, \infty[$ and one periodic on the variable $\hat{x}$.

- The overlapping zone is defined by $\{x \in \Omega, \eta < \mathrm{dist}(x,\Gamma) < 2\eta\}$. In this zone expansions (9.1.3) and (9.1.2) are assumed to be both valid and then should be equivalent.

We now built a system of equations in order to determine the sequences $(E^n, H^n, \underline{\hat{E}}^n, \underline{\hat{H}}^n)$.

9.1.1 The far field

Identifying formally each coefficient of (8.1.1) and the radiating condition (8.1.3) as an individual equation yields that for all $n > 0$ that:

$$\mathbf{rot} E^n = -ikH^n = 0 \quad \text{and} \quad \begin{cases} \mathbf{rot} H^0 = ikE^0 + J_{\text{source}}, \\ \mathbf{rot} H^n = ikE^n \text{ if } n > 0. \end{cases} \tag{9.1.5}$$

The fields E^n and H^n are required to satisfy the radiating condition (8.1.3). Thanks to the time harmonic Maxwell scattering theory(see [58]), we get that a necessary and sufficient missing information as this time to construct the field E^n is the knowledge of the tangential trace $\gamma_t E^n$. However we will see later that this last quantity is linked with the asymptotic behavior of the near field thanks to the matching condition.

9.1.2 The near field

For technical reasons of computation, we prefer to replace the expansion (9.1.3) by the following one:

$$\begin{cases} E^\delta(x) = \sum_{n\in\mathbb{N}} \delta^n \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1} \hat{E}^n(x_\Gamma; \hat{x}, \hat{\nu}), \\ H^\delta(x) = \sum_{n\in\mathbb{N}} \delta^n \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1} \hat{H}^n(x_\Gamma; \hat{x}, \hat{\nu}). \end{cases} \tag{9.1.6}$$

Here for all $n \in \mathbb{N}$, $\hat{E}_n$ and $\hat{H}_n$ are function defined on $\Gamma \times \hat{\Omega}$ and one periodic on $\hat{x}$. These series are required to formally satisfy the following equation:

$$\mathbf{rot} E^\delta = -ik\mu^\delta H^\delta \quad \text{and} \quad \mathbf{rot} H^\delta = ik\mu^\delta E^\delta, \tag{9.1.7}$$

with the boundary condition: For all $n \in \mathbb{N}$ if $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \partial\hat{\Omega}$ then:

$$\hat{E}_n(x_\Gamma; \hat{x}, \hat{\nu}) \times n(x_\Gamma) = 0 \quad \text{and} \quad \hat{\mu}(x_\Gamma; \hat{x}, \hat{\nu}) \hat{H}_n(x_\Gamma; \hat{x}, \hat{\nu}) \cdot n(x_\Gamma) = 0, \tag{9.1.8}$$

The detail of the analysis of this last expansion will be given later.

9.1.3 Matching condition

We will later see that the formal analysis of the required equation (9.1.7) implies for all $n \in \mathbb{N}$ the existence of polynomials $(P_E^n, P_h^n) \in C^\infty(\Gamma; \mathbb{C}_n[\hat{\nu}])^2$ such that:

$$\hat{E}^n(x_\Gamma; \hat{x}, \hat{\nu}) \underset{\hat{\nu}\to\infty}{\sim} P_E^n(x_\Gamma; \hat{\nu}) \quad \text{and} \quad \hat{H}^n(x_\Gamma; \hat{x}, \hat{\nu}) \underset{\hat{\nu}\to\infty}{\sim} P_H^n(x_\Gamma; \hat{\nu}), \tag{9.1.9}$$

where $(x_\Gamma, \hat{x}, \hat{\nu})$ are defined by (9.1.4). For all $0 \leq j \leq n$, $P_E^{n,j}$ and $P_H^{n,j}$ are respectively the j-th coefficient of the polynomial P_E^n and P_H^n. By inspiring from [37], we impose that:

$$P_E^{n+j,i}(x_\Gamma) = \frac{\partial_\nu^i \tilde{E}^n(x_\Gamma, 0)}{i!} \quad \text{and} \quad P_H^{n+j,i}(x_\Gamma) = \frac{\partial_\nu^i \tilde{H}^n(x_\Gamma, 0)}{i!}. \tag{9.1.10}$$

Here, the functions $(\tilde{E}^n, \tilde{H}^n)$ are defined for $(x'_\Gamma, \nu') \in \Gamma \times]-\delta, 2\eta[$ by:

$$\tilde{E}^n(x'_\Gamma, \nu') := \big(I + \nu' \mathcal{R}_\Gamma(x'_\Gamma)\big) E_n(x') \quad \text{and} \quad \tilde{H}^n(x'_\Gamma, \nu') := \big(I + \nu' \mathcal{R}_\Gamma(x'_\Gamma)\big) H_n(x'),$$

where we defined $x' := x'_\Gamma + \nu' n(x'_\Gamma)$. The equalities (9.1.10) are what we call the matching conditions. Let us recall now the reason of these relations: From (9.1.6) we have:

$$\begin{cases} \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big) E^\delta(x) = \displaystyle\sum_{n=0}^{\infty} \delta^n \hat{E}^n(x_\Gamma; \hat{x}, \hat{\nu}), \\ \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big) H^\delta(x) = \displaystyle\sum_{n=0}^{\infty} \delta^n \hat{H}^n(x_\Gamma; \hat{x}, \hat{\nu}). \end{cases}$$

We recall that in the matching zone we have $\nu > \eta$: Therefore from this, (9.1.4) and (9.1.1), we get that $\hat{\nu}$ tends to infinity when δ tends to zero. Therefore we can use the expansions (9.1.9):

$$\begin{cases} \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big) E^\delta(x) \approx \displaystyle\sum_{n,i} P_E^{n,i} \delta^n \hat{\nu}^i = \sum_{n,i} P_E^{n,i} \delta^{n-j} \nu^i, \\ \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big) H^\delta(x) \approx \displaystyle\sum_{n,i} P_H^{n,i} \delta^n \hat{\nu}^i = \sum_{n,i} P_H^{n,i} \delta^{n-j} \nu^i. \end{cases} \tag{9.1.11}$$

Moreover, ν tends to zero in the matching zone. Thus we can use the Taylor expansion formula:

$$\begin{cases} \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big) E^\delta\big(x_\Gamma + \nu n(x_\Gamma)\big) \approx \displaystyle\sum_{n,i} \frac{\partial_\nu^i \tilde{E}^n(x_\Gamma, 0)}{i!} \delta^n \nu^i \\ \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big) H^\delta\big(x_\Gamma + \nu n(x_\Gamma)\big) \approx \displaystyle\sum_{n,i} \frac{\partial_\nu^i \tilde{H}^n(x_\Gamma, 0)}{i!} \delta^n \nu^i. \end{cases}$$

Finally, identifying each coefficient power of δ and ν of these last expansions and the expansions in (9.1.11) yields the matching condition (9.1.10).

9.2 Analysis of the formal equation (9.1.7)

Here we will formally show that (9.1.7) implies a relation between the quantities $(\hat{E}^n, \hat{E}^n)$ and the previous quantities $(\hat{E}^{n-1}, \hat{E}^{n-1}) \cdots (\hat{E}^0, \hat{E}^0)$ that we will impose for all $n \in \mathbb{N}$ for further.

9.2.1 Curl of ψ_Γ $-\delta$-periodic field E^δ.

Here, we seek an expression of $\mathbf{rot}(E^\delta)$ where:

- For all $x \in \Omega^\delta$: $E^\delta(x) = \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1} \hat{E}(x_\Gamma; \hat{x}, \hat{\nu})$ where $(x_\Gamma, \hat{x}, \hat{\nu})$ are defined by (9.1.4).
- $\hat{E}$ is an element of $C^\infty\big(\Gamma; H_{\text{loc}}(\mathbf{rot}; \hat{\Omega})\big)$. We recall that:

$$H_{\text{loc}}(\mathbf{rot}; \hat{\Omega}) := \left\{ H : \hat{\Omega} \mapsto \mathbb{C}^3,\ \chi u \in H(\mathbf{rot}, \hat{\Omega}), \forall \chi \in \mathcal{D}\left(\overline{\hat{\Omega}}\right) \right\}.$$

That will be the object of Proposition 9.2.3. To state and prove this result we need to introduce the following notation:

Definition 9.2.1. *Let $u : \Gamma \mapsto \mathbb{C}^3$ be a vector field. We says that u is a tangential field if for all x_Γ we have $u(x_\Gamma) \in T_{x_\Gamma}\Gamma$ where $T_{x_\Gamma}\Gamma$ is the tangent space of x_Γ at the point x_Γ.*

We also need to introduce differential operators on the surface Γ and the infinite strip $\hat{Y}_\infty$:

- The operator $\mathbf{rot}_\Gamma$ is defined for all $\forall(x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ by:

$$\mathbf{rot}_\Gamma(\hat{E})(x_\Gamma; \hat{x}, \hat{\nu}) := \vec{\mathbf{rot}}_\Gamma(\hat{E}(x_\Gamma; \hat{x}, \hat{\nu}) \cdot n(x_\Gamma)) + \mathrm{rot}_\Gamma(\hat{E}^\Gamma)(x_\Gamma; \hat{x}, \hat{\nu}) n(x_\Gamma),$$

  with $\hat{E}^\Gamma(x_\Gamma; \hat{x}, \hat{\nu}) := \hat{E}(x_\Gamma; \hat{x}, \hat{\nu}) - \hat{E}(x_\Gamma; \hat{x}, \hat{\nu}) \cdot n(x_\Gamma)$. The vectorial surface rotational $\vec{\mathbf{rot}}_\Gamma$ and the scalar rotational rot_Γ are surface tangential differential operators that only concerns the variable x_Γ. They are defined for tangential u_Γ and scalar $\mathbf{u}_\Gamma$ fields defined on Γ by:

$$\mathrm{rot}_\Gamma(\mathbf{u}_\Gamma) := \mathrm{div}_\Gamma(n \times \mathbf{u}_\Gamma) \quad \text{and} \quad \vec{\mathbf{rot}}_\Gamma(u_\Gamma) := n \times \nabla_\Gamma(\mathrm{u}_\Gamma).$$

  We recall that the surface gradient ∇_Γ is defined for $u : \Gamma \mapsto \mathbb{R}$ and $x_\Gamma \in \Gamma$ by:

$$\nabla_\Gamma\, u(x_\Gamma) := \nabla \tilde{u}(x_\Gamma),$$

  where $\tilde{u}$ is an extension of u on C^δ such that for all $(x_\Gamma, \nu) \in \Gamma \times]-\delta, 0[$ we have $u(x_\Gamma) = \tilde{u}\big(x_\Gamma + n(x_\Gamma)\nu\big)$. The surface divergence is the unique differential operator such that for all tangential field $u_\Gamma : \Gamma \mapsto \mathbb{R}^3$ and $v : \Gamma \mapsto \mathbb{R}$:

$$\int_\Gamma \mathrm{div}_\Gamma(u_\Gamma) v dx_\Gamma = -\int u_\Gamma \cdot \nabla_\Gamma\, v dx_\Gamma.$$

- The operator $\hat{\mathbf{rot}}$ is a three dimensional differential operator that only concerns the variable $\hat{x}$ and $\hat{\nu}$. This operator is defined for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ by:

– If $x_\Gamma \in \Gamma_{\rm M}$ then:

$$\hat{\mathbf{rot}}(\hat{E})(x_\Gamma; \hat{x}, \hat{\nu}) := \Sigma_\Gamma(x_\Gamma) \cdot \Big(\mathbf{rot}_{\hat{x},\hat{\nu}}\big(\Sigma_\Gamma^\dagger(x_\Gamma) \cdot \hat{E}(x_\Gamma; \hat{x}, \hat{\nu})\big)\Big), \tag{9.2.12}$$

where we defined the matrix Σ_Γ for $x_\Gamma \in \Gamma_{\rm M}$ by:

$$\Sigma_\Gamma(x_\Gamma) := \sqrt{\det \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n^\dagger(x_\Gamma) \end{pmatrix}} \big(\mathrm{D}\,\psi_\Gamma{}^{-1}(x_\Gamma),\ n(x_\Gamma)\big). \tag{9.2.13}$$

We emphasize that $\Sigma_\Gamma(x_\Gamma)$ is well defined because we assumed that ψ_Γ is a diffeomorphism at the point x_Γ. We recall that for all smooth map $\psi : \Gamma \mapsto \mathbb{C}^d$ for some $d \in \mathbb{N}$, $\mathrm{D}\,\psi(x_\Gamma) : T_{x_\Gamma}\Gamma \mapsto \mathbb{C}^d$ is the differential defined for $v \in T_{x_\Gamma}\Gamma$ by:

$$\mathrm{D}\,\psi(x_\Gamma) := \mathrm{D}\,\big(\psi \circ \phi\big)(0)\big(\mathrm{D}\,\phi(0)\big)^{-1} v,$$

in the right hand-side of this equality D is the classical differential(the matrix containing the partial derivatives). Here $\phi : V(0) \mapsto V(x_\Gamma)$ is a local parameterization of Γ and $V(0) \subset \mathbb{R}^2$ and $V(x_\Gamma) \subset$ are respectively neighborhood of 0 and x_Γ.

– If $x_\Gamma \notin \Gamma_{\rm M}$ then:

$$\hat{\mathbf{rot}}(\hat{E})(x_\Gamma; \hat{x}, \hat{\nu}) := n(x_\Gamma) \times \partial_{\hat{\nu}} \hat{E}(x_\Gamma; \hat{x}, \hat{\nu}). \tag{9.2.14}$$

We need to recall a new notation:

Definition 9.2.2. *Let $\hat{u} : \Gamma \mapsto P(\hat{\Omega})$ be a reference function. We say that $\hat{u}$ is patching-ψ_Γ-admissible if for all $x_\Gamma \notin \Gamma_{\rm M}$ the function $\hat{u}(x_\Gamma; \cdot)$ only depends on the argument $\hat{\nu}$ i.e.*

$$\forall x_\Gamma \notin \Gamma_{\rm M},\ \exists \hat{u}_{\hat{\nu}}(x_\Gamma; \cdot) :]-1, 0[\mapsto \mathbb{R} \quad st \quad \forall (\hat{x}, \hat{\nu}) \in \mathbb{R}^2 \times]-1, 0[\quad \hat{u}(x_\Gamma; \hat{x}, \hat{\nu}) = \hat{u}_{\hat{\nu}}(x_\Gamma; \hat{\nu}).$$

Proposition 9.2.3. *If $\hat{E}$ is patching admissible then for all $x \in C_{\delta,\eta}$ we have:*

$$\mathbf{rot}(E^\delta)(x) = \det\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1}\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)\big(\delta^{-1}\hat{\mathbf{rot}}(\hat{E}(x_\Gamma; \hat{x}, \hat{\nu}) + \mathbf{rot}_\Gamma(\hat{E})(x_\Gamma; \hat{x}, \hat{\nu})\big),$$

where $(x_\Gamma, \hat{x}, \hat{\nu})$ are defined by (9.1.4).

Before giving the proof of this result, we introduce the map $\mathcal{L}$ defined for x near enough the boundary Γ by $\mathcal{L}(x) = (x_\Gamma, \nu)$ where (x_Γ, ν) are the unique solution of $x = x_\Gamma + \nu n(x_\Gamma)$.

Proof. We only give the proof for $x_\Gamma \in \Gamma_{\rm M}$ because the extension to $x_\Gamma \in \Gamma_{\rm M}$ is easy. We will use the change variable formula for the curl operator in [58]. First we introduce the transformation $\mathcal{L}_{\psi_\Gamma} : \Omega^\delta \mapsto \psi_\Gamma(\Gamma) \times]-\delta, \eta[$ defined for $x \in \Omega^\delta$ by:

$$\mathcal{L}_{\psi_\Gamma}(x) := \big(\psi_\Gamma(x_\Gamma), \nu\big) \quad \text{with} \quad (x_\Gamma, \nu) := \mathcal{L}(x),$$

and the vector field $R^\delta : \psi_\Gamma(\Gamma) \times]-\delta, \eta[\mapsto \mathbb{R}^3$ by $F^\delta := \mathrm{D}\,\mathcal{L}_{\psi_\Gamma}^{-\dagger} \circ \mathcal{L}_{\psi_\Gamma}^{-1} \cdot E^\delta \circ \mathcal{L}_{\psi_\Gamma}^{-1}$. Applying the result [58, Corollary 3.58] with this last vector field and the transformation $\mathcal{L}_{\psi_\Gamma}$ that :

$$\mathbf{rot}(E^\delta) = \det(\mathrm{D}\,\mathcal{L}_{\psi_\Gamma})\,\mathrm{D}\,\mathcal{L}_{\psi_\Gamma}^{-1} \cdot \mathbf{rot}\big(F^\delta\big), \tag{9.2.15}$$

Moreover using the formula of differential of the map $\mathcal{L}$ yields:

$$\mathrm{D}\,\mathcal{L}_{\psi_\Gamma}\circ\mathcal{L}^{-1} = \begin{pmatrix}\mathrm{D}\,\psi_\Gamma\\ n^\dagger\end{pmatrix}\cdot\big(I+\nu\mathcal{R}_\Gamma\big)^{-1} \quad\text{and}\quad \mathrm{D}\,\mathcal{L}_{\psi_\Gamma}^{-\dagger}\circ\mathcal{L}^{-1} = \begin{pmatrix}\mathrm{D}\,\psi_\Gamma{}^{-\dagger}\\ n^\dagger\end{pmatrix}\big(I+\nu\mathcal{R}_\Gamma\big). \quad (9.2.16)$$

Therefore for all $(x_r,\nu)\in\psi_\Gamma(\Gamma)\times]-\delta,\eta[$ we have:

$$F^\delta(x_r,\nu) := \hat{F}\left(x_r;\frac{x_r}{\delta},\frac{\nu}{\delta}\right),$$

where we defined $\hat{F}:\psi_\Gamma(\Gamma)\times\hat{Y}_\infty\mapsto\mathbb{R}^3$ for $(x_r,\hat{x},\hat{\nu})\in\psi_\Gamma(\Gamma)\times\hat{Y}_\infty\mapsto\mathbb{R}^3$ by:

$$\hat{F}(x_r;\hat{x},\hat{\nu}) := \mathrm{D}\,\psi_\Gamma{}^{-\dagger}\left(x_\Gamma\right)\cdot\hat{E}^\Gamma\left(x_\Gamma;\hat{x},\hat{\nu}\right)+e_3\left(\hat{E}\left(x_\Gamma;\hat{x},\hat{\nu}\right),n(x_\Gamma)\right)\quad\text{with}\quad x_\Gamma:=\psi_\Gamma{}^{-1}(x_r).$$

Thus we have for all $(x_r,\nu)\in\psi_\Gamma(\Gamma)\times]-\delta,\eta[$ that:

$$\mathbf{rot}(F^\delta)(x_r,\nu) = \mathbf{rot}_{x_r}\big(\hat{F}(x_r;\hat{x},\hat{\nu})\big)+\delta^{-1}\mathbf{rot}_{\hat{x},\hat{\nu}}\big(\hat{F}(x_r;\hat{x},\hat{\nu})\big)\quad\text{with}\quad(\hat{x},\hat{\nu}):=\frac{(x_r,\nu)}{\delta}.$$

On the one hand combining this last equation with the definition of the operator $\hat{\mathbf{rot}}$:

$$\mathbf{rot}_{\hat{x},\hat{\nu}}\big(\hat{F}(x_r;\hat{x},\hat{\nu})\big) = \mathbf{rot}_{\hat{x},\hat{\nu}}\Big(\big(\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^{-1},\ n\big)^\dagger\cdot\hat{E}(x_\Gamma;\hat{x},\hat{\nu})\Big)\quad\text{with}\quad x_\Gamma:=\psi_\Gamma{}^{-1}(x_r)$$

yields:

$$\det\big(\,\psi_\Gamma(x_\Gamma)\big)\big(\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^{-1},\ n\big)\cdot\mathbf{rot}_{\hat{x},\hat{\nu}}\big(\hat{F}(x_r;\hat{x},\hat{\nu})\big) = \hat{\mathbf{rot}}(\hat{E})(x_\Gamma;\hat{x},\hat{\nu}). \qquad (9.2.17)$$

On the other hand since the map $\psi_\Gamma:\Gamma\mapsto\mathbb{R}$ is locally a chart then we have the following expression of the operator rot_Γ and $\vec{\mathbf{rot}}_\Gamma$ for all tangential and scalar field $\hat{E}^\Gamma$ and $\hat{E}_\nu$(see [60]):

$$\begin{cases}\mathrm{rot}_\Gamma\hat{E}_\Gamma = \det\begin{pmatrix}\mathrm{D}\,\psi_\Gamma(x_\Gamma)\\ n^\dagger(x_\Gamma)\end{pmatrix}\mathrm{rot}_{\mathbb{R}^2}(\mathrm{D}\,\psi_\Gamma{}^{-\dagger}\,\hat{E}_\Gamma\circ\psi_\Gamma{}^{-1})\circ\psi_\Gamma,\\ \vec{\mathbf{rot}}_\Gamma\hat{E}_\nu = \det\begin{pmatrix}\mathrm{D}\,\psi_\Gamma(x_\Gamma)\\ n^\dagger(x_\Gamma)\end{pmatrix}\big(\,\mathrm{D}\,\psi_\Gamma{}^{-1},\ n(x_\Gamma)\big)\cdot\vec{\mathbf{rot}}_{\mathbb{R}^2}(\hat{E}_\nu\circ\psi_\Gamma{}^{-1})\circ\psi_\Gamma,\end{cases}$$

which leads to $\det\big(\,\psi_\Gamma(x_\Gamma)\big)\big(\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)^{-1},\ n\big)\mathbf{rot}_{x_r}\big(\hat{F}(x_r;\hat{x},\hat{\nu})\big) = \mathbf{rot}_\Gamma(\hat{E})(x_\Gamma;\hat{x},\hat{\nu})$. Combining this last equation with (9.2.16), (9.2.15) and (9.2.17) concludes the proof. □

9.2.2 Equation of the near field

Thanks to Proposition 9.2.3, (9.1.7) formally becomes for all $n\in\mathbb{N}$:

$$\hat{\mathbf{rot}}\big(\hat{E}^n\big) = f_n^E\quad\text{and}\quad\hat{\mathbf{rot}}\big(\hat{H}^n\big) = f_n^H, \qquad (9.2.18)$$

where we defined:

- The two following right hand-side:

$$\begin{cases} f_n^E = -\mathbf{rot}_\Gamma\big(\hat{E}^{n-1}\big)+ik\hat{\mu}\sum_{i=1}^{n}\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{H}^{n-i},\\ f_n^H = -\mathbf{rot}_\Gamma\big(\hat{H}^{n-1}\big)-ik\hat{\epsilon}\sum_{i=1}^{n}\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{E}^{n-i}.\end{cases}$$

- The map $\mathcal{M} : \Gamma \times \mathbb{R} \mapsto \mathcal{L}(\mathbb{R}^3)$ for $(x_\Gamma, \nu) \in \Gamma \times \mathbb{R}$ by:

$$\mathcal{M}(x_\Gamma, \nu) := \det\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)\,\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-2}. \tag{9.2.19}$$

- For all $x_\Gamma \in \Gamma$ and $i \in \mathbb{N}$ the quantity $\mathcal{M}_i(x_\Gamma) := \dfrac{\partial_\nu^i \mathcal{M}(x_\Gamma, 0)}{i!}$.

In (9.2.18) we have taken the convention $\hat{E}^{-1}(x_\Gamma; \hat{x}, \hat{\nu}) = \hat{H}^{-1}(x_\Gamma; \hat{x}, \hat{\nu}) = 0$. We only give the detail for the first line of this system because the computations for the second line are similar. Indeed from (9.1.6), we have for x near the boundary:

$$E^\delta(x) = \sum_{n \in \mathbb{N}} \delta^n \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1} \hat{E}^n(x_\Gamma; \hat{x}, \hat{\nu}).$$

Therefore by applying Proposition 9.2.3, we formally have:

$$\mathbf{rot}(E^\delta)(x) = \sum_{n \in \mathbb{N}} \delta^n \det\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1}\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)\big(\delta^{-1}\hat{\mathbf{rot}}(\hat{E}^n)(x_\Gamma; \hat{x}, \hat{\nu}) + \mathbf{rot}_\Gamma(\hat{E}^n)(x_\Gamma; \hat{x}, \hat{\nu})\big).$$

By using the definition (9.2.19) of the function $\mathcal{M}$ and (9.1.7), this becomes

$$ik\hat{\mu}(x_\Gamma; \hat{x}, \hat{\nu}) Q^\delta(x_\Gamma; \hat{x}, \hat{\nu}) = \sum_{n \in \mathbb{N}} \delta^n \big(\hat{\mathbf{rot}}(\hat{E}^n)(x_\Gamma; \hat{x}, \hat{\nu}) + \mathbf{rot}_\Gamma(\hat{E}^{n-1})(x_\Gamma; \hat{x}, \hat{\nu})\big), \tag{9.2.20}$$

where we defined:

$$Q^\delta(x_\Gamma; \hat{x}, \hat{\nu}) := \mathcal{M}(x_\Gamma, \nu)\big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1} H^\delta(x)$$

Moreover from the definition of the quantities $\mathcal{M}_j(x_\Gamma)$, we formally have:

$$\mathcal{M}(x_\Gamma, \nu) = \sum_{j=0}^{\infty} \nu^j \mathcal{M}_j(x_\Gamma) = \sum_{j=0}^{\infty} \delta^j \hat{\nu}^j \mathcal{M}_j(x_\Gamma).$$

Combining this with (9.1.6), yields:

$$Q^\delta(x_\Gamma; \hat{x}, \hat{\nu}) = \sum_{n \in \mathbb{N}} \delta^n \mathcal{M}_j(x_\Gamma) \hat{\nu}^n \hat{H}^n(x_\Gamma; \hat{x}, \hat{\nu}).$$

Combining this with (9.2.20) yields:

$$\sum_{n \in \mathbb{N}} \delta^n \big(\hat{\mathbf{rot}}(\hat{E}^n)(x_\Gamma; \hat{x}, \hat{\nu}) + \mathbf{rot}_\Gamma(\hat{E}^{n-1})(x_\Gamma; \hat{x}, \hat{\nu})\big) = \sum_{n \in \mathbb{N}} \delta^n \mathcal{M}_j(x_\Gamma) \hat{\nu}^n \hat{H}^n(x_\Gamma; \hat{x}, \hat{\nu}).$$

Identifying each power δ^n in this last equation conclude the proof of the first line of (9.2.18).

9.3 Explicit construction of a solution of the equation of the ansatz

Here we will give a process of construction of the sequence $(E^n, H^n, \hat{E}^n, \hat{H}^n)_{n\in\mathbb{N}}$ such that this sequence satisfies (9.1.5), (9.1.8), (9.1.9), (9.1.10) and (9.2.18). Let us summarize this section:

1. We construct operators $\mathcal{S}_E$ and $\mathcal{S}_H$ defined for one periodic function on $\hat{x}$ function $(f, g) : \hat{\Omega} \mapsto \mathbb{C}^3 \times \mathbb{R}$ satisfying some compatibility conditions then $\mathcal{S}_E(f, g)$ and $\mathcal{S}_H(f, g)$ are respectively solution of the two following problems:

   - Find u_E with $u_E \times n = 0$ on $\partial\hat{\Omega}$ such that:

   $$\hat{\mathbf{rot}}(u_E) = f \quad \text{and} \quad \widehat{\mathrm{div}}(\epsilon u_E) = g \text{ in } \hat{\Omega}. \tag{9.3.21}$$

   - Find u_H with $\mu u_H \cdot n = 0$ on $\partial\hat{\Omega}$ such that:

   $$\hat{\mathbf{rot}}(u_H) = f \quad \text{and} \quad \widehat{\mathrm{div}}(\mu u_H) = g \text{ in } \hat{\Omega}, \tag{9.3.22}$$

   in order to solve the system (9.2.18). We recall that the operator $\widehat{\mathrm{div}}$ is given for $u : \Gamma \times \hat{Y}_\infty \mapsto \mathbb{R}^3$ and $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ by:

   - If $x_\Gamma \in \Gamma_\mathrm{M}$ then:

   $$\widehat{\mathrm{div}}(u)(x_\Gamma; \hat{x}, \hat{\nu}) := \mathrm{div}_{\hat{x},\hat{\nu}} \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) u_\Gamma \\ n(x_\Gamma) \cdot u \end{pmatrix} (x_\Gamma; \hat{x}, \hat{\nu}). \tag{9.3.23}$$

   - If $x_\Gamma \notin \Gamma_\mathrm{M}$ then:

   $$\widehat{\mathrm{div}}(u)(x_\Gamma; \hat{x}, \hat{\nu}) := \partial_{\hat{\nu}}\big(n(x_\Gamma) \cdot u\big)(x_\Gamma; \hat{x}, \hat{\nu}). \tag{9.3.24}$$

     Here we can directly see that one of necessary compatibility conditions is $\widehat{\mathrm{div}}(f) = 0$. We will see after that a second necessary conditions are required.

2. We will give the values of the divergence $\widehat{\mathrm{div}}\left(\hat{\epsilon}\hat{E}^n\right)$ and $\widehat{\mathrm{div}}\left(\hat{\epsilon}\hat{E}^n\right)$ such that we have the necessary conditions : $\widehat{\mathrm{div}}(f_E^{n+1}) = \widehat{\mathrm{div}}(f_H^{n+1}) = 0$ for the next step.

3. We will give the algorithm of construction of the sequence $(E^n, H^n, \hat{E}^n, \hat{H}^n)_{n\in\mathbb{N}}$.

4. We prove that this definition well satisfies the required equations (9.1.5), (9.1.8), (9.1.9), (9.1.10) and (9.2.18). Moreover we will summarize in Lemma 9.3.11 the required property to prove the error estimate.

9.3.1 Construction of the solver operator $\mathcal{S}_E$ and $\mathcal{S}_H$

9.3.1.1 Kernel of the electrostatic problem

We need to introduce the space:

$$\mathbb{H}\big(\hat{Y}_\infty\big) := \left\{ u \in H^1_{\mathrm{loc}}(\hat{\Omega}),\ \|u\|^2_{\mathbb{H}\left(\hat{Y}_\infty\right)} := \int_{\hat{Y}_\infty} |\nabla u|^2 d\hat{x} d\hat{\nu} + \left| \int_\Sigma u d\hat{x} \right|^2 < \infty \text{ and u is one periodic in } \hat{x} \right\},$$

where $\Sigma :=]0,1[^2 \times \{0\}$. We introduce the following subspace:

$$\mathbb{H}_0\big(\hat{Y}_\infty\big) := \left\{ u \in \mathbb{H}\big(\hat{Y}_\infty\big),\ u = 0 \text{ on } \partial\hat{\Omega} \right\},$$

and the norm of this space is defined for $u \in \mathbb{H}_0\big(\hat{Y}_\infty\big)$ by $\|u\|_{\mathbb{H}_0\left(\hat{Y}_\infty\right)} := \|\nabla u\|_{L^2(\hat{Y}_\infty)^3}$. We need to introduce the operator $\widehat{\nabla}$ defined for $u : \Gamma \times \hat{\Omega} \mapsto \mathbb{R}$ and $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \Omega$ by:

$$\widehat{\nabla}\, u(x_\Gamma; \hat{x}, \hat{\nu}) := \mathrm{D}\, \psi_\Gamma{}^\dagger\, \nabla_{\hat{x},\hat{\nu}} u(x_\Gamma; \hat{x}, \hat{\nu}) + n(x_\Gamma) \partial_{\hat{\nu}} u(x_\Gamma; \hat{x}, \hat{\nu}) \text{ if } x_\Gamma \in \Gamma_{\mathrm{M}},$$

and:

$$\widehat{\nabla}\, u(x_\Gamma; \hat{x}, \hat{\nu}) := n(x_\Gamma) \partial_{\hat{\nu}} u(x_\Gamma; \hat{x}, \hat{\nu}) \text{ else.}$$

Hence we now can introduce for $x_\Gamma \in \Gamma$ the function $w_\epsilon(x_\Gamma; \cdot) \in \mathbb{H}_0\big(\hat{Y}_\infty\big)$ which is the solution of what we call the first cell problem: Find $w^\epsilon(x_\Gamma; \cdot) \in \mathbb{H}_0\big(\hat{Y}_\infty\big)$ such that for all $v \in \mathbb{H}_0\big(\hat{Y}_\infty\big)$ we have:

$$\int_{\hat{Y}_\infty} \hat{\epsilon}(x_\Gamma; \hat{x}, \hat{\nu})\, \widehat{\nabla}\, w^\epsilon(x_\Gamma; \hat{x}, \hat{\nu}) \cdot \widehat{\nabla}\, v(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \int_\Sigma v(x_\Gamma; \hat{x}, 0) d\hat{x}. \tag{9.3.25}$$

Thus we now can define the vector $\mathcal{N}_E : \Gamma \mapsto L^2_{\mathrm{loc}}(\hat{\Omega})$ defined for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ by:

$$\mathcal{N}_E(x_\Gamma; \hat{x}, \hat{\nu}) := n(x_\Gamma) \mathbb{I}_{\hat{\nu}>0}(\hat{\nu}) + \widehat{\nabla}\, w_\epsilon(x_\Gamma; \hat{x}, \hat{\nu}). \tag{9.3.26}$$

Proposition 9.3.1. *The vector $\mathcal{N}_E$ is an element of the kernel of the electrostatic problem in the sense that we have:*

$$\hat{\mathbf{rot}}(\mathcal{N}_E) = 0 \quad \textit{and} \quad \widehat{\mathrm{div}}(\epsilon \mathcal{N}_E) = 0, \tag{9.3.27}$$

with the boundary condition $\mathcal{N}_E \times n = 0$ on $\Gamma \times \hat{\Omega}$. Moreover we have $\mathcal{N}_E - n \in C^\infty\Big(\Gamma; L^2(\hat{Y}_\infty)\Big)^3$.

Proof. **Proof of** (9.3.27). First prove that $\hat{\mathbf{rot}}(\mathcal{N}_E) = 0$ and $\mathcal{N}_E \times n = 0$ on $\Gamma \times \partial\hat{\Omega}$. Indeed this last vector can be rewritten in the following form:

$$\mathcal{N}_E = \widehat{\nabla}\, \big(\hat{\nu} \mathbb{I}_{\hat{\nu}>0} + w^\epsilon\big),$$

and using that $\hat{\mathbf{rot}} \cdot \widehat{\nabla} = 0$ concludes the proof of $\hat{\mathbf{rot}}(\mathcal{N}_E) = 0$. Moreover the function $\hat{\nu} \mathbb{I}_{\hat{\nu}>0} + w^\epsilon$ vanishes on $\Gamma \times \partial\hat{\Omega}$ which leads to $\widehat{\nabla}(\mathbb{I}_{\hat{\nu}>0} + w^\epsilon) \times n = 0$ on $\Gamma \times \partial\hat{\Omega}$.

Let $x_\Gamma \in \Gamma$. Let us show that $\widehat{\operatorname{div}}\left(\mathcal{N}_E \epsilon(x_\Gamma;\cdot)\right) = 0$ in $\hat{\Omega}$. Indeed, for all $\phi \in \mathcal{D}(\hat{Y}_\infty)$, we have from (9.3.25) that:

$$\begin{aligned}
\langle \widehat{\operatorname{div}}\left(\epsilon\mathcal{N}_E(x_\Gamma;\cdot)\right), \phi\rangle_{\mathcal{D}^\dagger(\hat{Y}_\infty)-\mathcal{D}(\hat{Y}_\infty)} &= \int_{\hat{Y}_\infty}\left(\epsilon\mathcal{N}_E\cdot\widehat{\nabla}\,\phi\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},\\
&= \int_{\hat{Y}_\infty}\epsilon\,\widehat{\nabla}\,w^\epsilon\cdot\widehat{\nabla}\,\phi(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} + \int_{\hat{Y}_+}\partial_{\hat{\nu}}\phi(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},\\
&= \cancel{\int_{\hat{Y}_\infty}\left(\epsilon\,\widehat{\nabla}\,w^\epsilon\cdot\widehat{\nabla}\,\phi\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}} - \cancel{\int_\Sigma\phi d\hat{x}(\hat{x},0)} = 0.
\end{aligned}$$

Therefore we succeed in proving that $\widehat{\operatorname{div}}\left(\mathcal{N}_E\epsilon(x_\Gamma;\cdot)\right) = 0$ in $\hat{Y}_\infty$. The extension to $\hat{\Omega}$ is a consequence of the periodicity on $\hat{x}$ of the map $\mathcal{N}_E(x_\Gamma;\cdot)$: The periodicity of this vector implies that the normal derivative are continuous on each $]0,1[\times\{m\}\times]-1,\infty[$ and $\{m\}\times]0,1[\times]-1,\infty[$ for all $m\in\mathbb{Z}$ and $\widehat{\operatorname{div}}\left(\mathcal{N}_E\epsilon(x_\Gamma;\cdot)\right) = 0$ in $\hat{Y}_\infty + l$ for all $l\in\mathbb{Z}^2$.

Proof of $\mathcal{N}_E - n \in C^\infty\left(\Gamma; L^2(\hat{Y}_\infty)\right)$. We have $\mathcal{N}_E - n = \widehat{\nabla}\,w_\epsilon$. Moreover, thanks to (8.2.4) and (8.2.5) we can prove by using similar argument as the scalar case that $w_\epsilon \in C^\infty\left(\Gamma;\mathbb{H}_0(\hat{Y}_\infty)\right)$. We can conclude from the definition of the space $\mathbb{H}_0(\hat{Y}_\infty)$. □

We have a reciprocal result:

Proposition 9.3.2. *Let $u\in C^\infty(\Gamma; L^2_{\mathrm{loc}}(\hat{\Omega}))$ one periodic on the variable $\hat{x}$ and patching admissible satisfying*

$$\widehat{\mathbf{rot}}(u) = 0 \quad \textit{and} \quad \widehat{\operatorname{div}}(\epsilon u) = 0 \quad \textit{and} \quad u\times n = 0 \textit{ on } \Gamma\times\partial\hat{\Omega},$$

then there exists $U\in C^\infty(\Gamma)$ such that $u = U\cdot\mathcal{N}_E$.

Proof. We can prove by using a method of decomposition on the basis $\left((\hat{x},\hat{\nu})\mapsto\exp(i2\pi\hat{x}\cdot l)\right)_{l\in\mathbb{Z}^2}$ existence of a patching admissible function $\phi\in C^\infty(\Gamma;\mathbb{H}_0(\hat{Y}_\infty))$ and $U\in C^\infty(\Gamma)$ such that we have: $u = \widehat{\nabla}\,\phi + Un$. This last identity can by rewritten in the following form:

$$u = \widehat{\nabla}\left(\underbrace{\phi + (\hat{\nu}+1)\mathbb{I}_{\hat{\nu}<0} + \mathbb{I}_{\hat{\nu}>0}}_{\tilde{\phi}}\right) + U\mathbb{I}_{\hat{\nu}>0}n. \tag{9.3.28}$$

Moreover we can easily prove that:

$$\tilde{\phi}\in C^\infty\left(\Gamma;\mathbb{H}_0(\hat{Y}_\infty)\right). \tag{9.3.29}$$

Now we use $\widehat{\operatorname{div}}(\hat{\epsilon}u) = 0$. Let $\psi\in\mathbb{H}_0(\hat{Y}_\infty)$ and $x_\Gamma\in\Gamma$. Thanks to the periodicity of u and (9.3.29) we can apply the Green formula:

$$\begin{aligned}
0 &= \int_{\hat{Y}_\infty}\widehat{\operatorname{div}}(\hat{\epsilon}u)(x_\Gamma;\hat{x},\hat{\nu})\psi(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} = -\int_{\hat{Y}_\infty}\hat{\epsilon}(x_\Gamma;\hat{x},\hat{\nu})u(x_\Gamma;\hat{x},\hat{\nu})\cdot\widehat{\nabla}\,\psi(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},\\
&= -\int_{\hat{Y}_\infty}\hat{\epsilon}(x_\Gamma;\hat{x},\hat{\nu})u(x_\Gamma;\hat{x},\hat{\nu})\cdot\widehat{\nabla}\,\psi(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} - U(x_\Gamma)\int_{\hat{Y}_\infty}\partial_{\hat{\nu}}\psi d\hat{x}d\hat{\nu}.
\end{aligned}$$

Combining this with (9.3.28), yields:

$$\forall\psi\in\mathbb{H}_0(\hat{Y}_\infty),\ \int_{\hat{Y}_\infty}\hat{\epsilon}(x_\Gamma;\hat{x},\hat{\nu})\,\widehat{\nabla}\,\tilde{\phi}(x_\Gamma;\hat{x},\hat{\nu})\cdot\widehat{\nabla}\,\psi(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} = U(x_\Gamma)\int_\Sigma\psi(\hat{x},\hat{\nu})d\hat{x}.$$

Thanks to the uniqueness of the solution of (9.3.25) and the linearity of this problem, we directly deduce that $\tilde{\phi}(x_\Gamma;\hat{x},\hat{\nu}) = U(x_\Gamma)w^\epsilon(x_\Gamma;\hat{x},\hat{\nu})$. Combining this last identity with (9.3.28) conclude the proof of our proposition. $\square$

9.3.1.2 Kernel of the magneto-static problem

We introduce for $x_\Gamma \in \Gamma_{\mathrm{M}}$ and $i \in \{1,2\}$ the function $w_i(x_\Gamma;\hat{x},\hat{\nu}) \in \mathbb{H}\big(\hat{Y}_\infty\big)$ which is the unique of: Find $w_i(x_\Gamma;\hat{x},\hat{\nu}) \in \mathbb{H}\big(\hat{Y}_\infty\big)$ such that for all $v \in \mathbb{H}\big(\hat{Y}_\infty\big)$ we have:

$$\int_{\hat{Y}_\infty} \hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\,\widehat{\nabla}\, w_i(x_\Gamma;\hat{x},\hat{\nu})\cdot\widehat{\nabla}\, v(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu} = -\int_{\hat{Y}_\infty} \hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\partial_{\hat{x}_i} v(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}, \tag{9.3.30}$$

and to have uniqueness we impose that:

$$\int_\Sigma w_i(x_\Gamma)d\hat{x} = 0.$$

We introduce the map $\mathcal{N}_H : \Gamma\times\hat{Y}_\infty \mapsto \mathcal{L}(\mathbb{R}^3)$. This matrix is defined for $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma\times\hat{Y}_\infty$ by:

- If $x_\Gamma \notin \Gamma_{\mathrm{M}}$ then
$$\mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu}) := \mathbb{I}. \tag{9.3.31}$$
- If $x_\Gamma \in \Gamma_{\mathrm{M}}$ then for all $i \in \{1,2\}$:
$$\mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu})e_i(x_\Gamma) := e_i(x_\Gamma) + \widehat{\nabla}\, w_i \quad\text{and}\quad \mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu})n(x_\Gamma) = 0, \tag{9.3.32}$$
where $\big(e_i(x_\Gamma)\big)_{i=1,2}$ is defined for $i \in \{1,2\}$ by:
$$e_i(x_\Gamma) := \mathrm{D}\,\psi_\Gamma(x_\Gamma)^{-1}\hat{e}_i, \tag{9.3.33}$$
and $(\hat{e}_i)_i$ is the canonical basis of $\mathbb{R}^2$.

Proposition 9.3.3. *For all $v_\Gamma \in C^\infty(\Gamma;\mathbb{R}^3)$ tangential field, the vector $\mathcal{N}_H v_\Gamma$ is well an element of the kernel of the magneto-static problem in the sense that we have:*

$$\hat{\mathbf{rot}}\big(\mathcal{N}_H v_\Gamma\big) = 0 \quad \textit{and} \quad \widehat{\mathrm{div}}\,\big(\mu(\mathcal{N}_H v_\Gamma)\big) = 0,$$

with the boundary condition $\hat{\mu}(\mathcal{N}_H v_\Gamma)\cdot n = 0$ on $\Gamma\times\partial\hat{\Omega}$. Moreover we have:

$$\mathcal{N}_H v_\Gamma - v_\Gamma \in C^\infty\left(\Gamma; L^2(\hat{Y}_\infty)\right). \tag{9.3.34}$$

Proof. Let $v_\Gamma \in C^\infty(\Gamma;\mathbb{R}^3)$ be tangential field. The result is trivial for $x_\Gamma \notin \Gamma_{\mathrm{M}}$. Let $x_\Gamma \in \Gamma$. The property $\hat{\mathbf{rot}}(\mathcal{N}_H v_\Gamma) = 0$ is a direct consequence of the definition (9.3.31), (9.3.32) and $\hat{\mathbf{rot}}\,\widehat{\nabla} = 0$.

Now let us prove that we have $\widehat{\mathrm{div}}\,\big(\hat{\mu}(\mathcal{N}_H v_\Gamma)\big)(x_\Gamma;\cdot) = 0$. Let $\psi \in \mathcal{D}(\hat{Y}_\infty)$ then for all $i \in \{1,2\}$ we have:

$$\begin{aligned}\langle\widehat{\mathrm{div}}\,\big(\hat{\mu}(\mathcal{N}_H e_i)\big)(x_\Gamma;\hat{x},\hat{\nu}),\psi\rangle_{\mathcal{D}(\hat{Y}_\infty)^\dagger-\mathcal{D}(\hat{Y}_\infty)} &= -\int_{\hat{Y}_\infty}\Big(\hat{\mu}(\mathcal{N}_H e_i)\cdot\widehat{\nabla}\,\psi\Big)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},\\ &= -\int_{\hat{Y}_\infty}\Big(\hat{\mu}\big(\widehat{\nabla}\,w_i + e_i\big)\cdot\widehat{\nabla}\,\psi\Big)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}.\end{aligned}$$

Combining this with (9.3.33) yields:

$$\langle \widehat{\operatorname{div}}\left(\hat{\mu}(\mathcal{N}_H e_i)\right)(x_\Gamma;\hat{x},\hat{\nu}),\psi\rangle_{\mathcal{D}(\hat{Y}_\infty)^\dagger-\mathcal{D}(\hat{Y}_\infty)} = -\int_{\hat{Y}_\infty}\left(\hat{\mu}\,\widehat{\nabla}\, w_i\cdot\widehat{\nabla}\,\psi\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu},$$
$$+\int_{\hat{Y}_\infty}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\partial_{\hat{x}_i}\psi(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}.$$

Thanks to (9.3.30) yields for all $i\in\{1,2\}$ that $\widehat{\operatorname{div}}(\hat{\mu}\mathcal{N}_H e_i)(x_\Gamma;\cdot)=0$. Moreover $v_\Gamma(x_\Gamma)$ is a linear combination of $e_i(x_\Gamma)$ which conclude the proof of $\widehat{\operatorname{div}}(\hat{\mu}\mathcal{N}_H v_\Gamma)(x_\Gamma;\cdot)=0$ in $\hat{Y}_\infty$. The argument for the extension to $\hat{\Omega}$ are the same as the one of Proposition 9.3.1.

Now let us prove that $\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})u(x_\Gamma;\hat{x},\hat{\nu})\cdot n(x_\Gamma)=0$ on $\Gamma\times\hat{\Omega}$. Indeed from (9.3.30), we can prove that $\widehat{\nabla}\,w_i(x_\Gamma;\hat{x},\hat{\nu})\cdot n(x_\Gamma)=0$. Moreover since v_Γ is a tangential field, we have $v_\Gamma(x_\Gamma)\cdot n(x_\Gamma)=0$. Combining these two properties with (9.3.32) conclude the proof.

The proof of (9.3.34) takes the same argument than the one of Proposition 9.3.1. □

We have a reciprocal result:

Proposition 9.3.4. *Let $u\in C^\infty\big(\Gamma;L^2_{\mathrm{loc}}(\hat{Y}_\infty)\big)$ patching admissible satisfying*

$$\hat{\mathbf{rot}}(u)=0\quad\textit{and}\quad\widehat{\operatorname{div}}(\hat{\mu}u)=0\quad\textit{and}\quad\hat{\mu}u\cdot n=0\ \textit{on}\ \Gamma\times\partial\hat{\Omega},$$

then there exists a tangential field $U_\Gamma\in C^\infty(\Gamma;\mathbb{R}^3)$ such that $u=\mathcal{N}_H U_\Gamma$.

Proof. From $\hat{\mathbf{rot}}(u)=0$ and $\widehat{\operatorname{div}}(\hat{\mu}u)=0$, we can prove by using technique of decomposition on the function $\big((\hat{x},\hat{\nu})\mapsto\exp(i2\pi\hat{x}\cdot l)\big)_{l\in\mathbb{Z}^2}$ the existence of a function $\phi\in C^\infty(\Gamma;\mathbb{H}(\hat{Y}_\infty))$ patching admissible, a tangent vector field $v_\Gamma\in C^\infty(\Gamma;\mathbb{R}^3)$

$$u=\widehat{\nabla}\,\phi+v_\Gamma. \tag{9.3.35}$$

Thus it remains to prove that:

$$\widehat{\nabla}\,\phi(x_\Gamma;\hat{x},\hat{\nu})=\sum_i\widehat{\nabla}\,v^i_\Gamma(x_\Gamma)w_i(x_\Gamma;\hat{x},\hat{\nu})\text{ if }x_\Gamma\in\Gamma_{\mathrm{M}}\quad\text{and}\quad\widehat{\nabla}\,\phi(x_\Gamma;\hat{x},\hat{\nu})=0\text{ else }. \tag{9.3.36}$$

Here, if $x_\Gamma\in\Gamma_{\mathrm{M}}$, $\big(v^i_\Gamma(x_\Gamma)\big)_i$ are the unique scalar such that: $v_\Gamma(x_\Gamma)=\sum_i v^i_\Gamma(x_\Gamma)e_i(x_\Gamma)$. Indeed, let $\psi\in\mathbb{H}(\hat{Y}_\infty)$. Thanks to the condition $\widehat{\operatorname{div}}(\hat{\mu}u)(x_\Gamma;\cdot)=0$, $\hat{\mu}(x_\Gamma;\cdot)u(x_\Gamma;\cdot)\cdot n(x_\Gamma)=0$ and the periodicity property, we can apply Green formula:

$$0=\int_{\hat{Y}_\infty}\left(\hat{\mu}\,\widehat{\nabla}\,\phi,\widehat{\nabla}\,\psi\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}+\sum_i v^i_\Gamma(x_\Gamma)\int_{\hat{Y}_\infty}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\partial_{\hat{x}_i}\phi(\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}.$$

If $x_\Gamma\notin\Gamma_{\mathrm{M}}$ then this becomes:

$$\forall\psi,\ \mathbb{H}(\hat{Y}_\infty),\ 0=\int_{\hat{Y}_\infty}\left(\hat{\mu}\,\widehat{\nabla}\,\phi,\widehat{\nabla}\,\psi\right)(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}.$$

Hence $\widehat{\nabla}\,\phi=0$. If $x_\Gamma\in\Gamma_{\mathrm{M}}$ then we can use the use the uniqueness of the solution and the linearity of the problem (9.3.30). Therefore we conclude the proof of (9.3.36). Thus the proof is concluded. □

9.3.1.3 Construction of a right inverse $\mathcal{S}_E$ and $\mathcal{S}_H$ when the right hand-side satisfies the $\mathcal{P}^\infty$ property

Let $f : \Gamma\times\hat{\Omega} \mapsto \mathbb{R}^3$ and $g : \Gamma\times\hat{\Omega} \mapsto \mathbb{R}$ be two periodic in the variable $\hat{x}$. Here we construct two operator $\mathcal{S}_E$ and $\mathcal{S}_H$ such that under condition on f and g that $u_E := \mathcal{S}_E(f, g)$ and $u_H := \mathcal{S}_H(f, g)$ are respectively solution of (9.3.21) and (9.3.22). We first succeed to construct such operator when our right hand-side satisfies the $\mathcal{P}^\infty$:

Definition 9.3.5. *Let $u : \Gamma \mapsto L^2_{\text{loc}}(\hat{\Omega})$ one periodic on the variable $\hat{x}$. We say that u satisfies the $\mathcal{P}^\infty$ property if there exists $d \in \mathbb{N}$, a sequence $(u_l)_{l\in\mathbb{Z}^2\setminus\{0\}} \in C^\infty\big(\Gamma; \mathbb{C}_d[\hat{\nu}]\big)$ such that:*

$$\forall(x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_+,\ u(x_\Gamma; \hat{x}, \hat{\nu}) = \sum_{l\in\mathbb{Z}^2\setminus\{0\}} u_l(x_\Gamma; \hat{\nu})\phi_l(x_\Gamma; \hat{x}, \hat{\nu}),$$

where we defined the sequence of functions $(\phi_l)_{l\in\mathbb{Z}^2\setminus\{0\}}$ for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{Y}_+$ by:

$$\phi_l(x_\Gamma, \hat{x}, \hat{\nu}) := e^{i2\pi l\hat{x}} e^{-2\pi\lambda_l(x_\Gamma)\hat{\nu}} \quad \textit{with} \quad \lambda_l(x_\Gamma) := |\,\mathrm{D}\,\psi_\Gamma(x_\Gamma)l|.$$

Moreover, the sequence of polynomial are required to satisfies:

$$\forall q \in \mathbb{N}, \sum_{l\in\mathbb{Z}^2\setminus\{0\}} |l|^q \|u_l\|_{H^m(\Gamma;\mathbb{C}_d[\hat{\nu}])} < \infty.$$

We introduce for convenience the following vectorial space of function defined on $\Gamma \times \hat{Y}_\infty$:

$$\mathcal{F}(\Gamma\times\hat{Y}_\infty) := \left\{u \in C^\infty\left(\Gamma; L^2(\hat{Y}_\infty)\right), u \text{ is patching admissible and satisfies the } \mathcal{P}^\infty \text{ property}\right\}.$$

We first assume here that:

- The right hand-side of (9.3.21) belongs to:

$$\mathcal{F}_E(\Gamma \times \hat{Y}_\infty) := \left\{u \in \mathcal{F}(\Gamma \times \hat{Y}_\infty)^3, \widehat{\operatorname{div}}(u) = 0 \quad \text{and} \quad u \cdot n = 0 \text{ on } \Gamma \times \partial\hat{\Omega}\right\}.$$

- The right hand-side of (9.3.22) belongs to:

$$\mathcal{F}_H(\Gamma \times \hat{Y}_\infty) := \left\{u \in \mathcal{F}(\Gamma \times \hat{Y}_\infty)^3, \widehat{\operatorname{div}}(u) = 0\right\}.$$

Indeed, we have the following result:

Proposition 9.3.6. *There exist operators $\mathcal{S}_E$ and $\mathcal{S}_H$ defined for functions*

$$(f_E, f_H, g) \in \mathcal{F}_E(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}_H(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}(\Gamma \times \hat{Y}_\infty),$$

such that:

- *$u_E := \mathcal{S}_E(f_E, g)$ is a solution of (9.3.21). The map defined for $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ by:*

$$u_E(x_\Gamma; \hat{x}, \hat{\nu}) - \int_{\hat{Y}_-} f(x_\Gamma; \hat{x}, \hat{\nu}) \times n(x_\Gamma) d\hat{x}d\hat{\nu},$$

  belongs to the space $\mathcal{F}(\Gamma \times \hat{Y}_\infty)^3$.

- $u_H := \mathcal{S}_H(f, g)$ *is a solution of* (9.3.22). *The map defined for* $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ *by:*

$$u_H(x_\Gamma; \hat{x}, \hat{\nu}) - \left(\int_{\hat{Y}_-} g(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} \right) n(x_\Gamma),$$

belongs to the space $\mathcal{F}(\Gamma \times \hat{Y}_\infty)$.

We also need an intermediate result. It is easy to adapt the proof of [57, Theorem 3.20] to get the following generalization:
Proposition 9.3.7. *Let* $m \leq m_\Gamma$ *and two Hilbert spaces* E, F. *Then:*

$$\forall (A, u) \in C^{m_\Gamma}\big(\Gamma; \mathcal{L}(E, F)\big) \times H^m\big(\Gamma; E\big), \ Au \in H^m(\Gamma, F),$$

where $Au : \Gamma \mapsto F$ *is the map defined for* x_Γ *by* $A(x_\Gamma)u(x_\Gamma)$. *Moreover there exists* $C_m > 0$ *independent from* A *and* K *such that:*

$$\|Au\|_{H^m(\Gamma, F)} \leq C_m \|A\|_{C^{m_\Gamma}\left(\Gamma; \mathcal{L}(E,F)\right)} \cdot \|u\|_{H^m(\Gamma, E)}.$$

Proof of Proposition 9.3.6. We only give the proof for the operator $\mathcal{S}_E$. Let $x_\Gamma \in \Gamma$. Assume first that $x_\Gamma \notin \Gamma_{\mathrm{M}}$. Since f is patching admissible then we can construct $u_E(x_\Gamma; \cdot)$ that only depends on $\hat{\nu}$. In this case our equations are equivalent to:

$$n(x_\Gamma) \times \partial_{\hat{\nu}} u_E(x_\Gamma; \cdot) = f(x_\Gamma; \cdot) \quad \text{and} \quad \partial_{\hat{\nu}}\big(\hat{\epsilon}(x_\Gamma; \cdot) u_E \cdot n(x_\Gamma)\big) = g(x_\Gamma; \cdot).$$

We can easily see that this equation can be solved because we assumed that $f(x_\Gamma; \cdot) \cdot n(x_\Gamma) = 0$. The solution is given by:

$$u_E(x_\Gamma; \hat{x}, \hat{\nu}) := \int_{\hat{\nu}'=-1}^{\hat{\nu}} f(x_\Gamma; \hat{x}, \hat{\nu}') \times n(x_\Gamma) d\hat{\nu}' + n(x_\Gamma) \int_{\hat{\nu}'=0}^{\hat{\nu}} \hat{\epsilon}^{-1}(x_\Gamma; \hat{x}, \hat{\nu}') g(x_\Gamma; \hat{x}, \hat{\nu}') d\hat{\nu}'.$$

This last function well satisfies the boundary condition $u_E(x_\Gamma; \cdot) \times n(x_\Gamma) = 0$ on $\partial\hat{\Omega}$. We have:

$$u_E(x_\Gamma; \hat{x}, \hat{\nu}) - \int_{\hat{\nu}'=-1}^{0} f(x_\Gamma; \hat{x}, \hat{\nu}') \times n(x_\Gamma) d\hat{\nu}' = R(x_\Gamma; \hat{x}, \hat{\nu}),$$

where:

$$R(x_\Gamma; \hat{x}, \hat{\nu}) := \int_{\hat{\nu}'=0}^{\hat{\nu}} f(x_\Gamma; \hat{x}, \hat{\nu}') \times n(x_\Gamma) d\hat{\nu}' + n(x_\Gamma) \int_{\hat{\nu}'=0}^{\hat{\nu}} \hat{\epsilon}^{-1}(x_\Gamma; \hat{x}, \hat{\nu}') g(x_\Gamma; \hat{x}, \hat{\nu}') d\hat{\nu}'.$$

Since f and g satisfy the $\mathcal{P}^\infty$ then $f(x_\Gamma; \cdot)$ and $g(x_\Gamma; \cdot)$ vanishes on $\hat{Y}_+$. Thus the same holds for $R(x_\Gamma; \cdot)$. Thanks to the assumption $f \in C^\infty\big(\Gamma \setminus \overline{\Gamma_{\mathrm{M}}}; L^2(\hat{Y}_\infty)\big)^3$, $g \in C^\infty\big(\Gamma \setminus \overline{\Gamma_{\mathrm{M}}}; L^2(\hat{Y}_\infty)\big)$, (8.2.4) and (8.2.5), we deduce that $R \in C^\infty\big(\Gamma \setminus \overline{\Gamma_{\mathrm{M}}}; L^2(\hat{Y}_\infty)\big)^3$.

Now we investigate the case of $x_\Gamma \in \Gamma_{\mathrm{M}}$. Thanks to our assumption on ψ_Γ there exists an open subset ${\Gamma_{\mathrm{M}}}^\star$ of Γ such that $\overline{\Gamma_{\mathrm{M}}} \subset {\Gamma_{\mathrm{M}}}^\star$

$$\big(\Sigma_\Gamma, \Sigma_\Gamma^{-1}\big) \in C^\infty\big({\Gamma_{\mathrm{M}}}^\star; \mathcal{L}(\mathbb{R})^3\big),$$

where we recall that Σ_Γ is defined by (9.2.13). Thus we can introduce the following auxiliary functions:

$$\underline{u_E} := \Sigma_\Gamma^\dagger u_E,\ \underline{f} := \Sigma_\Gamma^{-1} f \quad \text{and} \quad \underline{\hat{\epsilon}} := \det \begin{pmatrix} \mathrm{D}\,\psi_\Gamma \\ n^\dagger \end{pmatrix} \Sigma_\Gamma^{-\dagger} \Sigma_\Gamma^{-1} \hat{\epsilon}.$$

Thanks to (9.3.23), (9.3.24), (9.2.12) and (9.2.14) our equations are equivalent to: Find $\underline{u_E}$ one periodic on $\hat{x}$ such that:

$$\mathbf{rot}\big(\underline{u_E}(x_\Gamma;\cdot)\big) = \underline{f},\ \operatorname{div}\big(\underline{\epsilon}(x_\Gamma;\cdot)\underline{u_E}(x_\Gamma;\cdot)\big) = g(x_\Gamma;\cdot) \quad \text{and} \quad \underline{u_E}(x_\Gamma;\cdot) \times e_3 = 0 \text{ on } \partial\hat{\Omega},$$

where $e_3 := \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\dagger$. Our new right hand-side $\underline{f}(x_\Gamma;\cdot)$ is well one periodic on $\hat{x}$ satisfies the $\mathcal{P}^\infty$ property and:

$$\operatorname{div}\big(\underline{f}(x_\Gamma;\cdot)\big) = 0 \quad \text{and} \quad \underline{f}(x_\Gamma;\cdot) \cdot e_3 \text{ on } \partial\hat{\Omega}.$$

Thus by inspiring from the planar 3D Maxwell case (see [35]), we can build a solution $\underline{u_E}^\star$ of:

$$\mathbf{rot}\big(\underline{u_E}^\star(x_\Gamma;\cdot)\big) = \underline{f},\ \operatorname{div}\big(\underline{u_E}^\star(x_\Gamma;\cdot)\big) = 0 \quad \text{and} \quad \underline{u_E}^\star(x_\Gamma;\cdot) \times e_3 = 0 \text{ on } \partial\hat{\Omega}, \tag{9.3.37}$$

and the map $\underline{R} : {\Gamma_{\mathrm{M}}}^\star \times \hat{\Omega} \mapsto \mathbb{R}^3$ defined for $x_\Gamma \in {\Gamma_{\mathrm{M}}}^\star$ by:

$$\underline{R}(x_\Gamma;\cdot) := \underline{u_E}^\star(x_\Gamma;\cdot) - \int_{\hat{Y}_-} \underline{f}(x_\Gamma;\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} \times e_3,$$

satisfies the $\mathcal{P}^\infty$ property and $\underline{R} \in C^\infty\big({\Gamma_{\mathrm{M}}}^\star; L^2(\hat{Y}_\infty)\big)$. However the field $\underline{u_E}$ does not yet satisfy $\operatorname{div}\big(\underline{u_E}^\star\big) = \underline{g}$. That is why we introduce the function $\underline{\phi}(x_\Gamma;\cdot)$ which is the unique solution of: Find $\underline{\phi}(x_\Gamma;\cdot) \in \mathbb{H}_0\big(\hat{Y}_\infty\big)$ such that:

$$\operatorname{div}\Big(\underline{\hat{\epsilon}(x_\Gamma;\cdot)}\nabla\underline{\phi}(x_\Gamma;\cdot)\Big) = -\operatorname{div}\Big(\underline{\hat{\epsilon}(x_\Gamma;\cdot)}\underline{u}_E^\star(x_\Gamma;\cdot)\Big) + \underline{g}(x_\Gamma;\cdot). \tag{9.3.38}$$

This problem is well posed because it is equivalent to the following variational formulation: Find $\underline{\phi}(x_\Gamma;\cdot) \in \mathbb{H}_0\big(\hat{Y}_\infty\big)$ such that:

$$A(x_\Gamma)\underline{\phi}(x_\Gamma;\hat{x},\hat{\nu}) = L(x_\Gamma) \text{ in } \mathbb{H}_0\big(\hat{Y}_\infty\big)^\dagger, \tag{9.3.39}$$

where $A(x_\Gamma)$ and $L(x_\Gamma)$ are defined for $(u,v) \in \mathbb{H}_0\big(\hat{Y}_\infty\big)^2$ by:

$$\langle A(x_\Gamma)\phi, \psi\rangle_{\mathbb{H}_0(\hat{Y}_\infty)^\dagger - \mathbb{H}_0(\hat{Y}_\infty)} := \int_{\hat{Y}_\infty} \big(\underline{\hat{\epsilon}}(x_\Gamma;\hat{x},\hat{\nu})\nabla\phi(\hat{x},\hat{\nu}), \nabla\psi(\hat{x},\hat{\nu})\big) d\hat{x} d\hat{\nu}$$

and

$$\langle L(x_\Gamma), \psi\rangle_{\mathbb{H}_0(\hat{Y}_\infty)^\dagger - \mathbb{H}_0(\hat{Y}_\infty)} := -\int_{\hat{Y}_\infty} \underline{R}(x_\Gamma;\hat{x},\hat{\nu}) \cdot \nabla\psi(\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} + \int_{\hat{Y}_\infty} g(x_\Gamma;\hat{x},\hat{\nu})\phi(\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu}.$$

The equation (9.3.39) is well posed because thanks to (8.2.4) we have that $A(x_\Gamma) : \mathbb{H}_0\big(\hat{Y}_\infty\big) \mapsto \mathbb{H}_0\big(\hat{Y}_\infty\big)^\dagger$ is invertible. Moreover, since g satisfies the $\mathcal{P}^\infty$, the function

$g(x_\Gamma;\cdot)$ decrease enough to have $\hat\nu g(x_\Gamma;\cdot) \in L^2(\hat Y_\infty)$. We can prove that it is a sufficient condition to have $g(x_\Gamma;\cdot) \in \mathbb{H}_0\big(\hat Y_\infty\big)^\dagger$. Moreover we have:

$$g \in C^\infty\big(\,\Gamma_{\mathrm{M}}{}^\star; \mathbb{H}_0\big(\hat Y_\infty\big)^\dagger\big). \tag{9.3.40}$$

Combining this with $R(x_\Gamma;\cdot) \in L^2(\hat Y_\infty)$ yields $L(x_\Gamma) \in \mathbb{H}_0\big(\hat Y_\infty\big)^\dagger$. Therefore (9.3.39) is well posed. We now define $\tilde u_E$ for $x_\Gamma \in \Gamma_{\mathrm{M}}{}^\star$ by:

$$\underline{u}_E(x_\Gamma;\cdot) := \underline{u}_E^\star(x_\Gamma;\cdot) + \nabla\underline{\phi}(x_\Gamma;\cdot). \tag{9.3.41}$$

Indeed, thanks to $\mathbf{rot}\nabla = 0$ and (9.3.37) we have:

$$\mathbf{rot}\underline{u}_E(x_\Gamma;\cdot) = \underline{f}(x_\Gamma;\cdot).$$

Combining (9.3.38) and $\underline{\phi}(x_\Gamma;\cdot) = 0$ on $\partial\hat\Omega$ with (9.3.37) yields:

$$\mathrm{div}\big(\underline{u_E}(x_\Gamma;\cdot)\big) = \tilde g(x_\Gamma;\cdot) \quad \text{and} \quad \underline{u_E}(x_\Gamma;\cdot) \times e_3 = 0 \text{ on } \partial\hat\Omega.$$

Now let us prove that the map $\mathcal{R} : \Gamma \times \hat\Omega$ defined for $x_\Gamma \in \Gamma_{\mathrm{M}}{}^\star$ by

$$\mathcal{R}(x_\Gamma;\cdot) := \underline{u}_E(x_\Gamma;\cdot) - \int_{\hat Y_-} \underline{f}(x_\Gamma;\hat x,\hat\nu)d\hat x d\hat\nu \times e_3,$$

belongs to $C^\infty\big(\,\Gamma_{\mathrm{M}}{}^\star; L^2(\hat Y_\infty)^3\big)$ and satisfies the $\mathcal{P}^\infty$ property. Indeed we first prove that:

$$\phi \in C^\infty\big(\,\Gamma_{\mathrm{M}}{}^\star; \mathbb{H}_0\big(\hat Y_\infty\big)\big). \tag{9.3.42}$$

Indeed, thanks to (8.2.4) we have:

$$\sup_{x_\Gamma \in \Gamma_{\mathrm{M}}{}^\star} \big\|A^{-1}(x_\Gamma)\big\|_{\mathcal{L}\left(\mathbb{H}_0\left(\hat Y_\infty\right)^\dagger, \mathbb{H}_0\left(\hat Y_\infty\right)\right)} < \infty. \tag{9.3.43}$$

Thanks to (8.2.5) we have $A \in C^\infty\Big(\,\Gamma_{\mathrm{M}}{}^\star; \mathcal{L}\big(\mathbb{H}_0\big(\hat Y_\infty\big), \mathbb{H}_0\big(\hat Y_\infty\big)^\dagger\big)\Big)$. Combining this with (9.3.43) yields $A^{-1} \in C^\infty\Big(\,\Gamma_{\mathrm{M}}{}^\star; \mathcal{L}\big(\mathbb{H}_0\big(\hat Y_\infty\big)^\dagger, \mathbb{H}_0\big(\hat Y_\infty\big)\big)\Big)$. Moreover thanks to:

$$\underline{R} \in C^\infty\big(\,\Gamma_{\mathrm{M}}{}^\star; L^2(\hat Y_\infty)^3\big),$$

and (9.3.40) we have $L \in C^\infty\Big(\,\Gamma_{\mathrm{M}}{}^\star; \mathbb{H}_0\big(\hat Y_\infty\big)^\dagger\Big)$. Therefore according to Proposition 9.3.7, we have $A^{-1}L \in C^\infty\Big(\,\Gamma_{\mathrm{M}}{}^\star; \mathbb{H}_0\big(\hat Y_\infty\big)\Big)$, which conclude the proof of (9.3.42).

Combining (9.3.42) with $\underline{R} \in C^\infty\Big(\,\Gamma_{\mathrm{M}}{}^\star; L^2(\hat Y_\infty)^3\Big)$ conclude the proof of:

$$\mathcal{R} \in C^\infty\Big(\,\Gamma_{\mathrm{M}}{}^\star; L^2(\hat Y_\infty)^3\Big). \tag{9.3.44}$$

Now let us prove that this last quantity satisfies the $\mathcal{P}^\infty$ property. Indeed, g and $\underline{u}_E^\star$ both satisfy and the $\mathcal{P}^\infty$ property. We refer the reader to the scalar case to get that it is a sufficient condition to have that $\nabla\phi$ satisfies the $\mathcal{P}^\infty$ property. Combining this with (9.3.41) and that $\underline{E}$ satisfies the $\mathcal{P}^\infty$ property conclude the proof. □

9.3.1.4 Extension of $\mathcal{S}_E$ and $\mathcal{S}_H$ in the polynomial space

Although Proposition 9.3.6 only yields the existence of suitable operators $\mathcal{S}_E$ and $\mathcal{S}_H$, we do not need for the sequel to have an explicit expression of these operators. Indeed, for the sequel, we will only use that these operator can solve (9.3.21), (9.3.22) and more precisely all the stated property in (9.3.6) to construct our ansatz. Thus hereafter $\mathcal{S}_E$ and $\mathcal{S}_H$ denote two operators satisfying the property appearing in Proposition 9.3.6.

Proposition 9.3.8. *Let* $(l_f^E, l_f^H, l_g) \in \mathcal{F}_E(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}_H(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}(\Gamma \times \hat{Y}_\infty)$*,* $d \in \mathbb{N}$ *and* $(p_f, p_g) \in C^\infty(\Gamma; \mathbb{C}_d[\hat{\nu}])^3 \times C^\infty(\Gamma; \mathbb{C}_d[\hat{\nu}])$ *satisfying:*

$$\forall x_\Gamma \in \Gamma,\ p_f(x_\Gamma; \cdot) \cdot n(x_\Gamma) = 0,$$

Define $f_E := l_f^E + p_f$*,* $f_H := l_f^H + p_f$ *and* $g = l_g + p_g$*. There exist extension of* $\mathcal{S}_E$ *and* $\mathcal{S}_H$ *for* (f_E, g) *and* (f_H, g) *such that:*

- *The function* $u_E := \mathcal{S}_E(f_E, g)$ *is a solution of* (9.3.22)*. There exist:*

  $$(p_E, l_E) \in \mathcal{F}(\Gamma \times \hat{Y}_\infty) \times C^\infty(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]),$$

  such that $u_E = p_E + l_E$*. For all* $x_\Gamma \in \Gamma$*,* $p_E(x_\Gamma; \cdot)$ *is the unique solution of:*

  $$\partial_{\hat{\nu}} p_E(x_\Gamma; \cdot) = n(x_\Gamma) \times p_f(x_\Gamma; \cdot) + p_g(x_\Gamma; \cdot) n(x_\Gamma), \tag{9.3.45}$$

  with the initial condition:

  $$p_E(x_\Gamma; 0) = n(x_\Gamma) \times \int_{\hat{Y}_-} f(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}. \tag{9.3.46}$$

- *The function* $u_H := \mathcal{S}_H(f_H, g)$ *is a solution of* (9.3.22)*. There exist:*

  $$(p_H, l_H) \in \mathcal{F}(\Gamma \times \hat{Y}_\infty) \times C^\infty(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]),$$

  such that $u_H = p_H + l_H$*. For all* $x_\Gamma \in \Gamma$*,* $p_H(x_\Gamma; \cdot)$ *is the unique solution of:*

  $$\partial_{\hat{\nu}} p_H(x_\Gamma; \cdot) = n(x_\Gamma) \times p_f(x_\Gamma; \cdot) + p_g(x_\Gamma; \cdot) n(x_\Gamma), \tag{9.3.47}$$

  with the initial condition:

  $$p_H(x_\Gamma; 0) = n(x_\Gamma) \int_{\hat{Y}_-} g(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}. \tag{9.3.48}$$

Proof. We emphasize that:

$$\mathcal{F}_E(\Gamma \times \hat{Y}_\infty) \cap C^\infty(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}])^3 = \mathcal{F}_H(\Gamma \times \hat{Y}_\infty) \cap C^\infty(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}])^3 = \{0\},$$

and

$$\mathcal{F}(\Gamma \times \hat{Y}_\infty) \cap C^\infty(\Gamma; \mathbb{C}_{d+1}[\hat{\nu}]) = \{0\}.$$

Therefore it remains to give a suitable definition of our operators for $(f, g) \in C^\infty(\Gamma; \mathbb{C}_d[\hat{\nu}])^3 \times C^\infty(\Gamma; \mathbb{C}_d[\hat{\nu}])$. We have:

$$\mathbb{I}_{\hat{\nu}<0} f \in \mathcal{F}_E(\Gamma \times \hat{Y}_\infty) \cap \mathcal{F}_H(\Gamma \times \hat{Y}_\infty) \quad \text{and} \quad \mathbb{I}_{\hat{\nu}<0} g \in \mathcal{F}(\Gamma \times \hat{Y}_\infty).$$

Therefore thanks to Proposition 9.3.6, we have:

$$\hat{\mathbf{rot}}(u_E^1) = \hat{\mathbf{rot}}(u_H^1) = f\mathbb{I}_{\hat{\nu}<0},\ \widehat{\mathrm{div}}\left(\hat{\epsilon}u_E^1\right) = \widehat{\mathrm{div}}\left(\hat{\mu}u_H^1\right) = g\mathbb{I}_{\hat{\nu}<0}, \tag{9.3.49}$$

and for all $x_\Gamma \in \Gamma$ the following boundary conditions:

$$u_E^1(x_\Gamma;\cdot) \times n(x_\Gamma) = 0 \quad \text{and} \quad u_H^1(x_\Gamma;\cdot) \cdot n(x_\Gamma) = 0 \text{ on } \partial\hat{\Omega}, \tag{9.3.50}$$

where we defined $u_E^1 := \mathcal{S}_E(f\mathbb{I}_{\hat{\nu}<0}g)$ and $u_H^1 := \mathcal{S}_E(f\mathbb{I}_{\hat{\nu}<0}g)$. Moreover the maps R_E and R_H defined for $x_\Gamma \in \Gamma$ by:

$$\begin{cases} R_E(x_\Gamma;\cdot) := u_E^1(x_\Gamma;\cdot) - \displaystyle\int_{\hat{Y}_-} f(x_\Gamma;\hat{x},\hat{\nu}) \times n(x_\Gamma) d\hat{x}d\hat{\nu}, \\ R_H(x_\Gamma;\cdot) := u_H^1(x_\Gamma;\cdot) - \left(\displaystyle\int_{\hat{Y}_-} g(x_\Gamma;\hat{x},\hat{\nu}) d\hat{x}d\hat{\nu}\right) n(x_\Gamma), \end{cases}$$

satisfy:

$$\big(R_E(x_\Gamma;\cdot), R_H(x_\Gamma;\cdot)\big) \in \mathcal{F}(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}(\Gamma \times \hat{Y}_\infty) \tag{9.3.51}$$

Define $p : \Gamma \mapsto \mathbb{C}_{d+1}[\hat{\nu}]$ for $x_\Gamma \in \Gamma$ by the unique solution of:

$$\partial_{\hat{\nu}} p(x_\Gamma;\cdot) = n(x_\Gamma) \times f(x_\Gamma;\cdot) + g(x_\Gamma;\cdot)n(x_\Gamma) \quad \text{and} \quad p(x_\Gamma;0) = 0. \tag{9.3.52}$$

Thanks to this we have:

$$\hat{\mathbf{rot}}\big(\mathbb{I}_{\hat{\nu}>0}p\big) = \mathbb{I}_{\hat{\nu}>0}f,\ \widehat{\mathrm{div}}\left(\hat{\epsilon}\mathbb{I}_{\hat{\nu}>0}p\right) = \mathbb{I}_{\hat{\nu}>0}g \quad \text{and} \quad \widehat{\mathrm{div}}\left(\hat{\mu}\mathbb{I}_{\hat{\nu}>0}p\right) = \mathbb{I}_{\hat{\nu}>0}g, \tag{9.3.53}$$

and it is trivial that $\mathbb{I}_{\hat{\nu}>0}p(x_\Gamma;\cdot) \times n(x_\Gamma) = 0$ and $\hat{\mu}(x_\Gamma;\cdot)\mathbb{I}_{\hat{\nu}>0}p(x_\Gamma;\cdot) \cdot n(x_\Gamma) = 0$ on $\partial\hat{\Omega}$. Combining this with (9.3.49) and (9.3.50) yields:

$$\hat{\mathbf{rot}}(u_E) = \hat{\mathbf{rot}}(u_H) = f,\ \widehat{\mathrm{div}}\left(\hat{\epsilon}u_E^1\right) = \widehat{\mathrm{div}}\left(\hat{\mu}u_H^1\right) = g, \tag{9.3.54}$$

and for all $x_\Gamma \in \Gamma$ the following boundary conditions:

$$u_E(x_\Gamma;\cdot) \times n(x_\Gamma) = 0 \quad \text{and} \quad u_H(x_\Gamma;\cdot) \cdot n(x_\Gamma) = 0 \text{ on } \partial\hat{\Omega}, \tag{9.3.55}$$

where we defined $u_E := u_E^1 + \mathbb{I}_{\hat{\nu}>0}p$ and $u_H := u_H^1 + \mathbb{I}_{\hat{\nu}>0}p$. Therefore defining $\mathcal{S}_E(f,g) := u_E$ and $\mathcal{S}_H(f,g) := u_H$ is a suitable choice.

Therefore it remains to prove that:

$$\big(u_E - p_E, u_H - p_H\big) \in \mathcal{F}(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}(\Gamma \times \hat{Y}_\infty). \tag{9.3.56}$$

Indeed, combining (9.3.45), (9.3.47), (9.3.46) and (9.3.48) with (9.3.52) yields for all $x_\Gamma \in \Gamma$ that:

$$\begin{cases} p_E(x_\Gamma;\cdot) = p(x_\Gamma;\cdot) + \displaystyle\int_{\hat{Y}_-} f(x_\Gamma;\hat{x},\hat{\nu}) \times n(x_\Gamma) d\hat{x}d\hat{\nu}, \\ p_H(x_\Gamma;\cdot) = p(x_\Gamma;\cdot) + \displaystyle\int_{\hat{Y}_-} g(x_\Gamma;\hat{x},\hat{\nu}) n(x_\Gamma) d\hat{x}d\hat{\nu}. \end{cases} \tag{9.3.57}$$

Therefore we have:

$$u_E - p_E = R_E - \mathbb{I}_{\hat{\nu}<0}p \quad \text{and} \quad u_H - p_H = R_H - \mathbb{I}_{\hat{\nu}<0}p.$$

Combining this with (9.3.51) conclude the proof of (9.3.56). □

We later see that the following result is required:
Proposition 9.3.9. *For all $s_\Gamma \in C^\infty(\Gamma;\mathbb{R})$ and $C^\infty(\Gamma;\mathbb{R}^3)$ tangential field*

$$\big(\mathcal{N}_E s_\Gamma - s_\Gamma n, \mathcal{N}_H v_\Gamma - v_\Gamma\big) \in \mathcal{F}(\Gamma \times \hat{Y}_\infty)^3 \times \mathcal{F}(\Gamma \times \hat{Y}_\infty)^3.$$

Proof. We have:

$$\mathcal{N}_E s_\Gamma - s_\Gamma n = \widehat{\nabla}\, w^\epsilon \quad \text{and} \quad v_\Gamma^i\, \widehat{\nabla}\, w_i,$$

where $(v_\Gamma^i)_i$ are C^∞ scalar field such that we have on Γ_{M} : $v_\Gamma = v_\Gamma^i e_i$. Therefore it remains to prove that $\widehat{\nabla}\, w^\epsilon$ and $(\widehat{\nabla}\, w_i)_i$ satisfy the $\mathcal{P}^\infty$ property. Indeed on the one hand we have seen that $w^\epsilon \in C^\infty\Big(\Gamma;\mathbb{H}_0\big(\hat{Y}_\infty\big)\Big)$ and $w_i \in C^\infty\Big(\Gamma;\mathbb{H}\big(\hat{Y}_\infty\big)\Big)$ for all $i \in \{1,2\}$. On the other hand thanks to (9.3.25) and (9.3.30) we have for all $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \mathbb{R}^2 \times \mathbb{R}_+^*$ that:

$$\forall i \in \{1,2\},\ \widehat{\operatorname{div}}\,\big(\widehat{\nabla}\, w_i\big)(x_\Gamma;\hat{x},\hat{\nu}) = \widehat{\operatorname{div}}\,\big(\widehat{\nabla}\, w^\epsilon\big)(x_\Gamma;\hat{x},\hat{\nu}) = 0,$$

and these function are one periodic on the variable $\hat{x}$. □

9.3.2 Values of the divergence $\widehat{\operatorname{div}}\,\big(\hat{\epsilon}\hat{E}^n\big)$ and $\widehat{\operatorname{div}}\,\big(\hat{\mu}\hat{H}^n\big)$

Thanks to (9.2.18) we know by induction the values of $\hat{\mathbf{rot}}\hat{E}_n$ and $\hat{\mathbf{rot}}\hat{H}^n$ with the knowledge of $\hat{E}^{n-1}\dots$ and $\hat{H}^{n-1}\dots$. According to Proposition 9.3.8, we need to impose a value for $\widehat{\operatorname{div}}(\hat{\epsilon}\hat{E}^n)$ and $\widehat{\operatorname{div}}(\hat{\mu}\hat{H}^n)$.

To give these values we need to introduce the surface divergence operator $\operatorname{div}_\Gamma$ defined for $u \in C^\infty(\Gamma;\mathbb{R}^3)$ and $x_\Gamma \in \Gamma$ by:

$$\operatorname{div}_\Gamma u(x_\Gamma) := \frac{1}{\sqrt{\det(\mathrm{D}\,\phi_x^\dagger(0)\cdot \mathrm{D}\,\phi_x(0))}}\operatorname{div}\left(\sqrt{\det(\mathrm{D}\,\phi_x^\dagger \cdot \mathrm{D}\,\phi_x)}\,\mathrm{D}\,\phi_x^{-1}\cdot u_\Gamma \circ \phi_x^{-1}\right)(0),$$

where $\phi_x : V(0) \mapsto V(x_\Gamma)$ is a local parameterization of Γ. Here $V(0) \subset \mathbb{R}^2$ and $V(x_\Gamma) \subset$ are respectively neighborhood of 0 and x_Γ.

We impose the following condition:

$$\widehat{\operatorname{div}}\,\big(\hat{\epsilon}\hat{E}^n\big) = g_n^E \quad \text{and} \quad \widehat{\operatorname{div}}\,\big(\hat{\mu}\hat{H}^n\big) = g_n^H, \tag{9.3.58}$$

where we defined for all $n' \in \mathbb{N}$:

$$\begin{cases} g_{n'}^E = -\displaystyle\sum_{i=1}^{n'} \operatorname{div}_\Gamma\big(\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{\epsilon}\hat{E}^{n'-i-1}\big) - \sum_{i=1}^{n'} \widehat{\operatorname{div}}\,\big(\mathcal{M}_i\hat{\nu}^i\epsilon\hat{E}^{n'-i}\big), \\ g_{n'}^H = -\displaystyle\sum_{i=1}^{n'} \operatorname{div}_\Gamma\big(\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{\mu}\hat{H}^{n'-i-1}\big) - \sum_{i=1}^{n'} \widehat{\operatorname{div}}\,\big(\mathcal{M}_i\hat{\nu}^i\mu\hat{H}^{n'-i}\big), \end{cases}$$

The reasons of this choice lies in the following form:
Proposition 9.3.10. *If we have :*

$$\widehat{\operatorname{div}}\,\big(\hat{\epsilon}\hat{E}^{n-1}\big) = g_{n-1}^E \quad \textit{and} \quad \widehat{\operatorname{div}}\,\big(\hat{\mu}\hat{H}^{n-1}\big) = g_{n-1}^H, \tag{9.3.59}$$

and

$$\hat{\mathbf{rot}}\big(\hat{E}^{n-1}\big) = f_{n-1}^E \quad \textit{and} \quad \hat{\mathbf{rot}}\big(\hat{H}^{n-1}\big) = f_{n-1}^H \tag{9.3.60}$$

then $\widehat{\operatorname{div}}(f_n^E) = \widehat{\operatorname{div}}(f_n^H) = 0$.

Proof. We have the following equality for all distribution u on $\Gamma \times \Omega$:

$$\widehat{\operatorname{div}}(\mathbf{rot}_\Gamma u) = -\operatorname{div}_\Gamma(\hat{\mathbf{rot}} u). \tag{9.3.61}$$

This last equality is a direct consequence of: For all $u \in C^\infty\big(\Gamma \times \hat{\Omega}; \mathbb{R}^3\big)$ and $\big(x_\Gamma, \hat{x}, \hat{\nu}\big) \in \Gamma_{\mathrm{M}} \times \hat{\Omega}$ we have that:

$$\begin{cases} \operatorname{div}_\Gamma(u)(x_\Gamma; \hat{x}, \hat{\nu}) := \det \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n^\dagger(x_\Gamma) \end{pmatrix} \operatorname{div}_{x_r} \left(\det \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n^\dagger(x_\Gamma) \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n^\dagger(x_\Gamma) \end{pmatrix} u(x_\Gamma; \hat{x}, \hat{\nu}) \right) \Bigg|_{x_r = \psi_\Gamma(x_\Gamma)}, \\ \widehat{\operatorname{div}}(u)(x_\Gamma; \hat{x}, \hat{\nu}) := \operatorname{div}_{\hat{x},\hat{\nu}} \left(\begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n(x_\Gamma) \end{pmatrix} u(x_\Gamma; \hat{x}, \hat{\nu}) \right), \\ \vec{\mathbf{rot}}_\Gamma(u)(x_\Gamma; \hat{x}, \hat{\nu}) := \det \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n^\dagger(x_\Gamma) \end{pmatrix} \big(\mathrm{D}\,\psi_\Gamma{}^{-1}(x_\Gamma),\ n(x_\Gamma) \big)\, \mathbf{rot}_{x_r} \left(\begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma)^{-\dagger} \\ n^\dagger(x_\Gamma) \end{pmatrix} u(x_\Gamma; \hat{x}, \hat{\nu}) \right) \Bigg|_{x_r := \psi_\Gamma(x} \\ \hat{\mathbf{rot}}(u)(x_\Gamma; \hat{x}, \hat{\nu}) := \det \begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma) \\ n^\dagger(x_\Gamma) \end{pmatrix} \big(\mathrm{D}\,\psi_\Gamma{}^{-1}(x_\Gamma),\ n(x_\Gamma) \big)\, \mathbf{rot}_{\hat{x},\hat{\nu}} \left(\begin{pmatrix} \mathrm{D}\,\psi_\Gamma(x_\Gamma)^{-\dagger} \\ n^\dagger(x_\Gamma) \end{pmatrix} u(x_\Gamma; \hat{x}, \hat{\nu}) \right), \end{cases}$$

and $\operatorname{div}_{\hat{x},\hat{\nu}} \mathbf{rot}_{x_r} + \operatorname{div}_{x_r} \hat{\mathbf{rot}}_{\hat{x},\hat{\nu}} = 0$. Thanks to (9.3.61) we have

$$\widehat{\operatorname{div}}(f_n^E) = \operatorname{div}_\Gamma \big(\hat{\mathbf{rot}} \hat{E}^{n-1} \big) + ik\, \widehat{\operatorname{div}} \left(\hat{\mu} \sum_{i=1}^{n} \mathcal{M}_{i-1} \hat{\nu}^{i-1} \hat{H}^{n-i} \right).$$

Combining this with (9.3.60), yields:

$$\widehat{\operatorname{div}}(f_n^E) = \cancel{-\operatorname{div}_\Gamma \Big(\mathbf{rot}_\Gamma \big(\hat{E}^{n-2} \big) \Big)} + ik \operatorname{div}_\Gamma \left(\hat{\mu} \sum_{i=1}^{n} \mathcal{M}_{i-1} \hat{\nu}^{i-1} \hat{H}^{n-i} \right) + ik\, \widehat{\operatorname{div}} \left(\hat{\mu} \sum_{i=1}^{n} \mathcal{M}_{i-1} \hat{\nu}^{i-1} \hat{H}^{n-1-i} \right)$$

Combining this with (9.3.59) concludes the proof of $\widehat{\operatorname{div}}(f_n^E) = 0$. The proof of $\widehat{\operatorname{div}}(f_n^H) = 0$ is exactly the same, which conclude the proof. $\square$

9.3.3 Algorithm of construction

The nears field $(\hat{E}^n, \tilde{H}_n)_{n\in\mathbb{N}}$ and the far field $(E_n, H_n)_{n\in\mathbb{N}}$ are recursively defined. These quantities are defined for $n = -1$ by zero. Now let us explain how we define for some $n \in \mathbb{N}$ the quantities $(\hat{E}^n, \hat{H}^n)$ and (E^n, H^n) from the knowledge of the quantities $(\hat{E}^{n'}, \hat{H}^{n'})_{0\leq n'\leq n-1}$ and $(E^{n'}, H^{n'})_{0\leq n'\leq n-1}$.

9.3.3.1 Right-hand-side of the cell problem

We recall for the convenience that:

$$\begin{cases} f_n^E := -\mathbf{rot}_\Gamma\big(\hat{E}^{n-1}\big) + ik\hat{\mu}\displaystyle\sum_{i=1}^{n}\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{H}^{n-i}, \\ f_n^H := -\mathbf{rot}_\Gamma\big(\hat{H}^{n-1}\big) - ik\hat{\epsilon}\displaystyle\sum_{i=1}^{n}\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{E}^{n-i}, \\ g_n^E := -\displaystyle\sum_{i=1}^{n}\mathrm{div}_\Gamma\left(\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{\epsilon}\hat{E}^{n-i-1}\right) + \widehat{\mathrm{div}}\left(\mathcal{M}_i\hat{\nu}^i\hat{\epsilon}\hat{E}^{n-i}\right), \\ g_n^H := -\displaystyle\sum_{i=1}^{n}\mathrm{div}_\Gamma\left(\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{\mu}\hat{H}^{n-i-1}\right) + \widehat{\mathrm{div}}\left(\mathcal{M}_i\hat{\nu}^i\hat{\mu}\hat{H}^{n-i}\right). \end{cases}$$

9.3.3.2 Definition of the far field

The far field (E_n, H_n) are defined by the unique solution of: Find $(E_n, H^n) \in H_{\mathrm{loc}}(\mathbf{rot};\Omega)^2$ such that:

$$\mathbf{rot}E^n = -ikH^n \qquad \text{and} \qquad \mathbf{rot}H^n = ikE^n + \delta_{n0}J_{\mathrm{source}}, \tag{9.3.62}$$

with the boundary condition for all x_Γ, $E^n(x_\Gamma)\times n(x_\Gamma) = \displaystyle\int_{\hat{Y}_-} f_n^E(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}$ and the far field have to satisfies the radiating condition (8.1.3).

9.3.3.3 Definition of the near field

- The electric near field $\hat{E}^n$ is defined for $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma\times\hat{\Omega}$ by:

$$\hat{E}^n(x_\Gamma;\hat{x},\hat{\nu}) := (E^n(x_\Gamma), n(x_\Gamma))\mathcal{N}_E(x_\Gamma;\hat{x},\hat{\nu}) + \big(\mathcal{S}_E(f_n^E, g_n^E)\big)(x_\Gamma;\hat{x},\hat{\nu}). \tag{9.3.63}$$

- The magnetic near field $\hat{H}^n$ is defined by:

$$\hat{H}^n(x_\Gamma;\hat{x},\hat{\nu}) := \mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu})H^n(x_\Gamma) + \big(\mathcal{S}_H(f_n^H, g_n^H)\big)(x_\Gamma;\hat{x},\hat{\nu}). \tag{9.3.64}$$

Here we prove that we have the following result of construction of the ansatz.

Lemma 9.3.11. *The nears field $(\hat{E}^n, \hat{H}^n)_{n\in\mathbb{N}}$ and the far field $(E^n, H^n)_{n\in\mathbb{N}}$ well satisfy the required property. More precisely we have for all $n \in \mathbb{N}$ that:*

- *The far field (E_n, H_n) belongs to $C^\infty(\overline{\Omega})$.*
- *The functions $\hat{E}^n$ and $\hat{H}^n$ take the following form:*

$$\hat{E}^n = P_n^E + R_n^E \quad \text{and} \quad \hat{H}^n = P_n^H + R_n^H, \tag{9.3.65}$$

  for some $(R_n^E, R_n^H) \in \mathcal{F}(\Gamma\times\hat{Y}_\infty)^2$ and $(P_E^n, P_H^n) \in C^\infty\big(\Gamma;\mathbb{C}_n[\hat{\nu}]\big)\times C^\infty\big(\Gamma;\mathbb{C}_n[\hat{\nu}]\big)$.

- *For all $\hat{E}^n(x_\Gamma;\cdot)\times n(x_\Gamma) = 0$ and $\hat{H}^n(x_\Gamma;\cdot)\cdot n(x_\Gamma) = 0$ on $\partial\hat{\Omega}$.*

- *The matching condition are satisfied:*

$$\forall 0 \le i \le n,\ P_E^{n,i} = \frac{\partial_\nu^i \tilde{E}^{n-i}(x_\Gamma, 0)}{i!} \quad \text{and} \quad P_H^{n,i} = \frac{\partial_\nu^i \tilde{H}^{n-i}(x_\Gamma, 0)}{i!}. \tag{9.3.66}$$

- *The equation (9.2.18) and (9.3.58) are satisfied.*

We prove this last lemma with an induction on the number n'. The result is trivial for $n' = -1$. Let $n' \in \mathbb{N}$ such that we have for all $n < n'$ such that all quantity an property appearing in Lemma 9.3.11 is well defined and true. We prove now that this implies that the same holds for $n' = n$.

Proposition 9.3.12. *There exists functions*

$$(l_{n'}^E, l_{n'}^H, d_{n'}^E, d_{n'}^H) \in \mathcal{F}_E(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}_H(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}(\Gamma \times \hat{Y}_\infty) \times \mathcal{F}(\Gamma \times \hat{Y}_\infty)$$

and polynomials

$$(F_{n'}^E, F_{n'}^H, G_{n'}^E, G_{n'}^H) \in C^\infty\big(\Gamma; \mathbb{C}_{n'-1}[\hat{\nu}]^3\big)^2 \times C^\infty\big(\Gamma; \mathbb{C}_{n'-1}[\hat{\nu}]\big)^2,$$

such that the following decompositions of our right hand-side hold:

$$f_{n'}^E = F_{n'}^E + l_{n'}^E,\ f_{n'}^H = F_{n'}^H + l_{n'}^H,\ g_{n'}^E = G_{n'}^E + d_{n'}^E \quad \text{and} \quad g_{n'}^H = G_{n'}^H + d_{n'}^H.$$

The polynomials $F_{n'}^E$, $F_{n'}^H$, $G_{n'}^E$ and $G_{n'}^H$ are given by:

$$\begin{cases} F_{n'}^E = -\partial_{\hat{\nu}}\big(n \times P_E^{n'-1}\big) - \mathbf{rot}_\Gamma(P_E^{n'-1}) - ik \displaystyle\sum_{j=0}^{n'-1} \mathcal{M}^j \hat{\nu}^j P_H^{n'-(j+1)}, \\ F_{n'}^H = -\partial_{\hat{\nu}}\big(n \times P_H^{n'-1}\big) - \mathbf{rot}_\Gamma(P_H^{n'-1}) + ik \displaystyle\sum_{j=0}^{n'-1} \mathcal{M}^j \hat{\nu}^j P_E^{n'-(j+1)}, \\ G_{n'}^E \cdot n = -\partial_{\hat{\nu}} \left(\displaystyle\sum_{j=1}^{n'} \mathcal{M}_j \hat{\nu}^j P_E^{n'-j} \cdot n \right) - \mathrm{div}_\Gamma \left(\displaystyle\sum_{j=1}^{n'} \mathcal{M}_{j-1} \hat{\nu}^{j-1} (n \times P_E^{n'-j}) \times n \right), \\ G_{n'}^H \cdot n = -\partial_{\hat{\nu}} \left(\displaystyle\sum_{j=1}^{n'} \mathcal{M}_j \hat{\nu}^j P_H^{n'-j} \cdot n \right) - \mathrm{div}_\Gamma \left(\displaystyle\sum_{j=1}^{n'} \mathcal{M}_{j-1} \hat{\nu}^{j-1} (n \times P_H^{n'-j}) \times n \right). \end{cases}$$

Proof. We refer the reader for the scalar case because the proof of Proposition 2.5.15 (See Chapter 2), have similar arguments. □

Proposition 9.3.13. *The right-hand-side $f_{n'}^E$ and $f_{n'}^H$ satisfy the compatibility conditions:*

$$\widehat{\mathrm{div}}(f_{n'}^E) = \widehat{\mathrm{div}}(f_{n'}^H) = 0 \quad \text{and} \quad p_{n'}^E \cdot n = p_{n'}^H \cdot n = 0. \tag{9.3.67}$$

Moreover we have on $\Gamma \times \partial\hat{\Omega}$ that:

$$f_{n'}^E \cdot n = 0. \tag{9.3.68}$$

Proof. The proof of $\widehat{\mathrm{div}}(f^E_{n'}) = \widehat{\mathrm{div}}(f^H_{n'}) = 0$ is already given in Proposition 9.3.10.

Projecting the equation (9.3.62) yields on the normal yields:

$$\forall 0 \le p \le n'-1 \begin{cases} 0 = \mathrm{rot}_\Gamma\big(n \times (\tilde{E}_{n'-p} \times n)\big) + ik\big(\mathcal{M}\tilde{H}_{n'-p}\cdot n\big) \\ 0 = \mathrm{rot}_\Gamma\big(n \times (\tilde{H}_{n'-p} \times n)\big) - ik\big(\mathcal{M}\tilde{E}_{n'-p}\cdot n\big). \end{cases}$$

and differentiate p times in the variable ν these last equations yields:

$$\forall 0 \le p \le n'-1 \begin{cases} 0 = \mathrm{rot}_\Gamma\big(n \times \partial^p_\nu(\tilde{E}_{n'-1-p} \times n)\big) + ik\displaystyle\sum_{p'=0}^{p}\binom{p}{p'} p'!\big(\mathcal{M}_{p'}\partial_\nu^{p-p'}\tilde{H}_{n'-1-p}\cdot n\big), \\ 0 = \mathrm{rot}_\Gamma\big(n \times \partial^p_\nu(\tilde{H}_{n'-1-p} \times n)\big) - ik\displaystyle\sum_{p'=0}^{p}\binom{p}{p'} p'!\big(\mathcal{M}_{p'}\partial_\nu^{p-p'}\tilde{E}_{n'-1-p}\cdot n\big). \end{cases}$$

Combining this last equality with the recurrence hypothesis (9.3.66) yields:

$$\forall 0 \le p \le n'-1 \begin{cases} 0 = \mathrm{rot}_\Gamma\big(n \times (P_E^{n'-1,p} \times n)\big) + ik\displaystyle\sum_{p'=0}^{p}\mathcal{M}_{p'}P_H^{n'-1-p',p-p'}\cdot n, \\ 0 = \mathrm{rot}_\Gamma\big(n \times (P_H^{n'-1,p} \times n)\big) - ik\displaystyle\sum_{p'=0}^{p}\mathcal{M}_{p'}P_E^{n'-1-p',p-p'}\cdot n, \end{cases}$$

which can be rewritten:

$$\forall 0 \le p \le n'-1 \begin{cases} 0 = \mathrm{rot}_\Gamma\big(n \times (P_E^{n'-1,p} \times n)\big) + ik\left(\displaystyle\sum_{p'=0}^{p}\mathcal{M}_{p'}\hat{\nu}^{p'}P_H^{n'-1-p'}\cdot n\right)_p, \\ 0 = \mathrm{rot}_\Gamma\big(n \times (P_H^{n'-1,p} \times n)\big) - ik\left(\displaystyle\sum_{p'=0}^{p}\mathcal{M}_{p'}\hat{\nu}^{p'}P_E^{n'-1-p'}\cdot n\right)_p, \end{cases}$$

Combining this last equation with Proposition 9.3.12 concludes the proof of (9.3.67).

The property (9.3.68) is due to the boundary condition on $\Gamma \times \partial\hat{\Omega}$ satisfied by the previous term $\hat{E}^{n'-1}\cdots\hat{E}^0$ and $\hat{H}^{n'-1}\cdots\hat{H}^0$. Indeed we have for all $(x_\Gamma;\hat{x},\hat{\nu}) \in \Gamma \times \partial\hat{\Omega}$:

$$\begin{aligned} f^E_{n'}(x_\Gamma;\hat{x},0)\cdot n &= \left(-\mathbf{rot}_\Gamma\big(\hat{E}^{n'-1}\big) + ik\hat{\mu}\sum_{i=1}^{n'}\mathcal{M}_{i-1}\hat{\nu}^{i-1}\hat{H}^{n'-i}\right)(x_\Gamma;\hat{x},0)\cdot n(x_\Gamma), \\ &= -\mathrm{div}_\Gamma\,\cancel{\big(\hat{E}^{n'-1}(x_\Gamma;\hat{x},0)\times n(x_\Gamma)\big)} + ik\hat{\mu}\cancel{\hat{H}^{n'-1}(x_\Gamma;\hat{x},0)\cdot n(x_\Gamma)} = 0. \end{aligned}$$

Corollary 9.3.14. *For $n = n'$ the equations (9.2.18) and (9.3.58) are well satisfied. There exist $(P_E^{n'}, P_H^{n'}, R_E^{n'}, R_H^{n'}) \in C^\infty(\Gamma;\mathbb{C}_{n'}[\hat{\nu}])^2 \times \mathcal{F}(\Gamma\times\hat{Y}_\infty)^2$ such that*

$$\hat{E}^{n'} = P_E^{n'} + R_E^{n'} \quad \text{and} \quad \hat{H}^{n'} = P_H^{n'} + R_H^{n'},$$

where we defined for all $(x_\Gamma,\hat{x},\hat{\nu}) \in \Gamma\times\hat{\Omega}$:

$$\begin{cases} \hat{E}^{n'}(x_\Gamma;\hat{x},\hat{\nu}) := (E_{n'}(x_\Gamma), n(x_\Gamma))\mathcal{N}_E(x_\Gamma;\hat{x},\hat{\nu}) + \big(\mathcal{S}_E(f^E_{n'}, g^E_{n'})\big)(x_\Gamma;\hat{x},\hat{\nu}), \\ \hat{H}^{n'}(x_\Gamma;\hat{x},\hat{\nu}) := \mathcal{N}_H(x_\Gamma;\cdot)H_{n'}(x_\Gamma) + \big(\mathcal{S}_H(f^H_{n'}, g^H_{n'})\big)(x_\Gamma;\hat{x},\hat{\nu}). \end{cases} \tag{9.3.69}$$

The coefficients are given for all $1 \le p \le n'$ by:

$$\begin{cases} \left(P_E^{n'} \times n\right)_p = -\dfrac{1}{p}\vec{\mathbf{rot}}_\Gamma\left(P_E^{n'-1,p-1}\cdot n\right)_p - ik\dfrac{1}{p}\left(\displaystyle\sum_{j=0}^{p-1}\mathcal{M}_j\left(n\times(P_H^{n'-1-j,p-1-j}\times n)\right)\right)_p, \\ \left(P_H^{n'} \times n\right)_p = -\dfrac{1}{p}\vec{\mathbf{rot}}_\Gamma\left(P_H^{n'-1,p-1}\cdot n\right)_p + ik\dfrac{1}{p}\displaystyle\sum_{j=0}^{p-1}\mathcal{M}_j\left(n\times(P_E^{n'-1-j,p-1-j}\times n)\right)_p, \\ (P_E^{n'}\cdot n)_p = -\displaystyle\sum_{j=1}^{p}\mathcal{M}_j(P_E^{n'-j,p-j}\cdot n) - \dfrac{1}{p}\sum_{j=0}^{p-1}\mathrm{div}_\Gamma\left(\mathcal{M}^j\left(n\times(P_E^{n'-1-j,p-1-j}\times n)\right)\right)_p, \\ (P_H^{n'}\cdot n)_p = -\displaystyle\sum_{j=1}^{p}\mathcal{M}_j(P_H^{n'-j,p-j}\cdot n) - \dfrac{1}{p}\sum_{j=0}^{p-1}\mathrm{div}_\Gamma\left(\mathcal{M}^j\left(n\times(P_H^{n'-1-j,p-1-j}\times n)\right)\right)_p, \end{cases} \tag{9.3.70}$$

and for $p = 0$ we have for all $x_\Gamma \in \Gamma$ that:

$$P_E^{n'}(x_\Gamma;0) = E^{n'}(x_\Gamma) \quad \textit{and} \quad P_H^{n'}(x_\Gamma;0) = H^{n'}(x_\Gamma). \tag{9.3.71}$$

Proof. Thanks to Proposition 9.3.12 and Proposition 9.3.13 we can apply Proposition 9.3.8 which leads that $\mathcal{S}_E(f_{n'}^E, g_{n'}^E)$ and $\mathcal{S}_H(f_{n'}^E, g_{n'}^E)$ are solution of (9.2.18) and (9.3.58) for $n = n'$. Moreover combining the definition (9.3.69) with Proposition 9.3.1 and Proposition 9.3.3 which conclude the proof of (9.2.18) and (9.3.58).

Applying Proposition 9.3.8 yields the existence of

$$(R_E^{n',0}, R_H^{n',0}, P_{E,0}^n, P_{H,0}^n) \in \mathcal{F}(\Gamma\times\hat{Y}_\infty)^2 \times C^\infty\left(\Gamma;\mathbb{C}_n[\hat{\nu}]\right)^2,$$

such that:

$$\mathcal{S}_E(f_{n'}^E, g_{n'}^E) = R_E^{n',0} + P_{E,0}^{n'} \quad \text{and} \quad \mathcal{S}_H(f_{n'}^H, g_{n'}^H) = R_H^{n',0} + P_{H,0}^{n'},$$

and the polynomials $(P_{E,0}^{n'}, P_{H,0}^{n'}) \in C^\infty\left(\Gamma;\mathbb{C}_{n'}[\hat{\nu}]\right)^2$ are the unique solutions of:

$$\partial_{\hat{\nu}}P_{E,0}^{n'} = G_{n'}^E \times n + G_{n'}^E n \quad \text{and} \quad \partial_{\hat{\nu}}P_{H,0}^{n'} = F_{n'}^H \times n + G_{n'}^H n, \tag{9.3.72}$$

with the initial conditions for all $x_\Gamma \in \Gamma$:

$$P_{E,0}^{n'}(x_\Gamma;0) = \int_{\hat{Y}_-} f_{n'}^E(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\times n(x_\Gamma) \quad \text{and} \quad P_{H,0}^{n'}(x_\Gamma;0) = \left(\int_{\hat{Y}_-} g_{n'}^H(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)n(x_\Gamma).$$

Thanks to the definition (9.3.69), Proposition 9.3.9, Proposition 9.3.1 and Proposition 9.3.3, there exist

$$(R_E^{n'}, R_H^{n'}, P_E^{n'}, P_H^{n'}) \in \mathcal{F}(\Gamma\times\hat{Y}_\infty)^2 \times C^\infty\left(\Gamma;\mathbb{C}_n[\hat{\nu}]\right)^2,$$

such that:

$$\hat{E}^{n'} = R_E^{n'} + P_E^{n'} \quad \text{and} \quad \hat{H}^{n'} = R_H^{n'} + P_H^{n'},$$

and $P_E^{n'}$ and $P_H^{n'}$ are the unique polynomials satisfying (9.3.72) and the initial conditions for all $x_\Gamma \in \Gamma$:

$$\begin{cases} P_E^{n'}(x_\Gamma;0) = \left(\displaystyle\int_{\hat{Y}_-} n(x_\Gamma)\times f_{n'}^E(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right) + E^{n'}(x_\Gamma)\cdot n(x_\Gamma)n(x_\Gamma), \\ P_H^{n'}(x_\Gamma;0) = \left(\displaystyle\int_{\hat{Y}_-} g_{n'}^H(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right)n(x_\Gamma) + n(x_\Gamma)\times\left(H^{n'}(x_\Gamma)\times n(x_\Gamma)\right). \end{cases} \tag{9.3.73}$$

Thus integrating one time on the variable $\hat{\nu}$ the equation (9.3.72) conclude the proof of (9.3.70). With computation we show that we have:

$$\mathrm{div}_\Gamma \left(\int_{\hat{Y}_-} f^E_{n'}(\cdot;\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} \right) = ik \int_{\hat{Y}_-} g^H_{n'}(\cdot;\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu},$$

and we recall that we thanks to the definition of the far field E^n that for all $x_\Gamma \in \Gamma$:

$$\int_{\hat{Y}_-} f^E_{n'}(x_\Gamma;\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} = E^{n'}(x_\Gamma) \times n(x_\Gamma). \tag{9.3.74}$$

These two last equalities lead to:

$$\forall x_\Gamma \in \Gamma,\ ik \int_{\hat{Y}_-} g^E_{n'}(x_\Gamma;\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} = \mathbf{rot}(E^{n'})(x_\Gamma) \cdot n(x_\Gamma).$$

Combining this with the Maxwell equations that E^n and H^n satisfy yields:

$$\forall x_\Gamma,\ \cancel{ik} \int_{\hat{Y}_-} g^E_{n'}(x_\Gamma;\hat{x},\hat{\nu}) d\hat{x} d\hat{\nu} = \cancel{ik} H^{n'}(x_\Gamma) \cdot n(x_\Gamma).$$

Combining this last equation and (9.3.74) with (9.3.73) concludes the proof of (9.3.71).□

Proposition 9.3.15. *The far field (E^n, H^n) defined by the unique solution of the problem (9.3.62) belongs to $C^\infty(\overline{K})$ for all open subset of Ω such that $\overline{K} \subset \Omega$.*

Proof. It is a direct consequence of regularity results of the Maxwell equation. □

Proposition 9.3.16. *We have for all $0 \le i \le n'$ that:*

$$\forall 0 \le i \le n',\ P_E^{n',i} = \frac{\partial^i_\nu \tilde{E}^{n'-i}(x_\Gamma, 0)}{i!} \quad \text{and} \quad P_H^{n',i} = \frac{\partial^i_\nu \tilde{H}^{n'-i}(x_\Gamma, 0)}{i!} \tag{9.3.75}$$

Proof. Using the PDE (9.3.62) yields:

$$\begin{cases} \partial_\nu(\tilde{E}^{n'-p} \times n) = -\vec{\mathbf{rot}}_\Gamma\big(\tilde{E}^{n'-p} \cdot n\big) - ik\mathcal{M}\big(n \times (\tilde{H}^{n'-p} \times n)\big), \\ \partial_\nu(\tilde{H}^{n'-p} \times n) = -\vec{\mathbf{rot}}_\Gamma\big(\tilde{H}^{n'-p} \cdot n\big) + ik\mathcal{M}\big(n \times (\tilde{E}^{n'-p} \times n)\big), \\ \partial_\nu(\mathcal{M} \cdot \tilde{E}^{n'-p}) = -\,\mathrm{div}_\Gamma \Big(\mathcal{M}\big(n \times (\tilde{E}^{n'-p} \times n)\big)\Big), \\ \partial_\nu(\mathcal{M} \cdot \tilde{H}^{n'-p}) = -\,\mathrm{div}_\Gamma \Big(\mathcal{M}\big(n \times (\tilde{H}^{n'-p} \times n)\big)\Big). \end{cases}$$

For all $p \in \mathbb{N} \setminus \{0\}$, differentiating these last equations $p-1$ times on the variable ν yields on $\Gamma \times \{0\}$:

$$\begin{cases} \partial^p_\nu(\tilde{E}^{n'-p} \times n) = -\vec{\mathbf{rot}}_\Gamma\big(\partial^{p-1}_\nu(\tilde{E}^{n'-p} \cdot n)\big) - ik \displaystyle\sum_{j=0}^{p-1} \binom{p-1}{j} j! \mathcal{M}_j \partial^{p-1-j}_\nu \big(n \times (\tilde{H}^{n'-p} \times n)\big), \\ \partial^p_\nu(\tilde{H}^{n'-p} \times n) = -\vec{\mathbf{rot}}_\Gamma\big(\partial^{p-1}_\nu(\tilde{H}^{n'-p} \cdot n)\big) + ik \displaystyle\sum_{j=0}^{p-1} \binom{p-1}{j} j! \mathcal{M}_j \partial^{p-1-j}_\nu \big(n \times (\tilde{E}^{n'-p} \times n)\big), \end{cases}$$

and

$$\begin{cases} \partial_\nu^p(\tilde{E}^{n'-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\binom{p}{j}\partial_\nu^{p-j}(j!\mathcal{M}_j\tilde{E}^{n'-p}\cdot n) - \sum_{j=0}^{p-1}\binom{p-1}{j}\operatorname{div}_\Gamma\Big(j!\mathcal{M}_j\partial_\nu^{p-1-j}\big(n\times(\tilde{E}^{n'-p}\times n)\big)\Big) \\ \partial_\nu^p(\tilde{H}^{n'-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\binom{p}{j}\partial_\nu^{p-j}(j!\mathcal{M}_j\tilde{H}^{n'-p}\cdot n) - \sum_{j=0}^{p-1}\binom{p-1}{j}\operatorname{div}_\Gamma\Big(j!\mathcal{M}_j\partial_\nu^{p-1-j}\big(n\times(\tilde{H}^{n'-p}\times n)\big)\Big) \end{cases}$$

For the sequel, all quantity that will appear will be evaluated in $(x_\Gamma, 0)$. We can rewrite the last lines in the following forms:

$$\begin{cases} \partial_\nu^p(\tilde{E}^{n'-p}\times n) = -\,\vec{\mathbf{rot}}_\Gamma\big(\partial_\nu^{p-1}(\tilde{E}^{n-1-(p-1)}\cdot n)\big) \\ \qquad\qquad - ik\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}j!\mathcal{M}_j\partial_\nu^{p-1-j}\big(n\times(\tilde{H}^{n-1-j-(p-1-j)}\times n)\big), \\ \partial_\nu^p(\tilde{H}^{n-p}\times n) = -\,\vec{\mathbf{rot}}_\Gamma\big(\partial_\nu^{p-1}(\tilde{H}^{n-1-(p-1)}\cdot n)\big) \\ \qquad\qquad + ik\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}j!\mathcal{M}_j\partial_\nu^{p-1-j}\big(n\times(\tilde{E}^{n-1-j-(p-1-j)}\times n)\big), \end{cases}$$

and

$$\begin{cases} \partial_\nu^p(\tilde{E}^{n-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\binom{p}{j}\partial_\nu^{p-j}(j!\mathcal{M}_j\tilde{E}^{n-j-(p-j)}\cdot n) \\ \qquad\qquad -\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}\operatorname{div}_\Gamma\Big(j!\mathcal{M}^j\partial_\nu^{p-1-j}\big(n\times(\tilde{E}^{n-1-j-(p-1-j)}\times n)\big)\Big), \\ \partial_\nu^p(\tilde{H}^{n-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\binom{p}{j}\partial_\nu^{p-j}(j!\mathcal{M}_j\tilde{H}^{n-j-(p-j)}\cdot n) \\ \qquad\qquad -\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}\operatorname{div}_\Gamma\Big(j!\mathcal{M}^j\partial_\nu^{p-1-j}\big(n\times(\tilde{H}^{n-1-j-(p-1-j)}\times n)\big)\Big). \end{cases}$$

Moreover injecting recurrence hypothesis (9.3.75) in these last equalities yields:

$$\begin{cases} \partial_\nu^p(\tilde{E}^{n-p}\times n) = -\,\vec{\mathbf{rot}}_\Gamma\big((p-1)!P_E^{n-1,p-1}\cdot n\big) \\ \qquad\qquad - ik\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}j!\mathcal{M}_j(p-1-j)!\big(n\times(P_H^{n-1-j,p-1-j}\times n)\big), \\ \partial_\nu^p(\tilde{H}^{n-p}\times n) = -\,\vec{\mathbf{rot}}_\Gamma\big((p-1)!P_H^{n-1,p-1}\cdot n\big) \\ \qquad\qquad + ik\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}j!\mathcal{M}_j(p-1-j)!\big(n\times(P_E^{n-1-j,p-1-j}\times n)\big), \end{cases}$$

and

$$\begin{cases} \partial_\nu^p(\tilde{E}^{n-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\binom{p}{j}(p-j)!(j!\mathcal{M}_j P_E^{n-j,p-j}\cdot n) \\ \qquad\qquad\qquad -\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}(p-1-j)!\,\mathrm{div}_\Gamma\left(j!\mathcal{M}^j\big(n\times(P_E^{n-1-j,p-1-j}\times n)\big)\right), \\ \partial_\nu^p(\tilde{H}^{n-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\binom{p}{j}(p-j)!(j!\mathcal{M}_j P_H^{n-j,p-j}\cdot n) \\ \qquad\qquad\qquad -\displaystyle\sum_{j=0}^{p-1}\binom{p-1}{j}(p-1-j)!\,\mathrm{div}_\Gamma\left(j!\mathcal{M}^j\big(n\times(P_H^{n-1-j,p-1-j}\times n)\big)\right), \end{cases}$$

Therefore thanks to the definition of binomial coefficient, we have for all $1\le p\le n$ that these last equalities simplify in the the following forms:

$$\begin{cases} \dfrac{1}{p!}\partial_\nu^p(\tilde{E}^{n-p}\times n) = -\dfrac{1}{p}\vec{\mathbf{rot}}_\Gamma\big(P_E^{n-1,p-1}\cdot n\big) - ik\dfrac{1}{p}\displaystyle\sum_{j=0}^{p-1}\mathcal{M}_j\big(n\times(P_H^{n-1-j,p-1-j}\times n)\big), \\ \dfrac{1}{p!}\partial_\nu^p(\tilde{H}^{n-p}\times n) = -\dfrac{1}{p}\vec{\mathbf{rot}}_\Gamma\big(P_H^{n-1,p-1}\cdot n\big) + ik\dfrac{1}{p}\displaystyle\sum_{j=0}^{p-1}\mathcal{M}_j\big(n\times(P_E^{n-1-j,p-1-j}\times n)\big), \\ \dfrac{1}{p!}\partial_\nu^p(\tilde{E}^{n-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\mathcal{M}_j(P_E^{n-j,p-j}\cdot n) - \dfrac{1}{p}\displaystyle\sum_{j=0}^{p-1}\mathrm{div}_\Gamma\left(\mathcal{M}^j\big(n\times(P_E^{n-1-j,p-1-j}\times n)\big)\right), \\ \dfrac{1}{p!}\partial_\nu^p(\tilde{H}^{n-p}\cdot n) = -\displaystyle\sum_{j=1}^{p}\mathcal{M}_j(P_H^{n-j,p-j}\cdot n) - \dfrac{1}{p}\displaystyle\sum_{j=0}^{p-1}\mathrm{div}_\Gamma\left(\mathcal{M}^j\big(n\times(P_H^{n-1-j,p-1-j}\times n)\big)\right), \end{cases}$$

Therefore combining these last lines with Corollary 9.3.14 conclude the proof of (9.3.75).□

Chapter 10

Theoretical justification of the asymptotic expansion

Let $n \in \mathbb{N}$. We construct global function on Ω^δ defined by:

$$E^n_{\eta,\delta} := (1-\chi_\eta)E^n_\delta + \chi_\eta \mathcal{I}_{\delta,\eta}(\hat{E}^n_\delta),$$

where:

- $E^n_\delta := \sum_{i=0}^{n} \delta^i E^i$ and $\hat{E}^n_\delta := \sum_{i=0}^{n} \delta^i \hat{E}^i$.
- The so called "scaling operator" $\mathcal{I}_{\delta,\eta}$ is defined for $\hat{E} : \Gamma \times \hat{\Omega} \mapsto \mathbb{C}^3$ by the map $\mathcal{I}_{\delta,\eta}\,\hat{E} : C_{\delta,\eta} \mapsto \mathbb{C}^3$. This map is defined for $x \in C_{\delta,\eta}$ by:

  $$\mathcal{I}_{\delta,\eta}\,\hat{E}(x) := \big(I + \nu \mathcal{R}_\Gamma(x_\Gamma)\big)^{-1} \hat{E}(x_\Gamma; \hat{x}, \hat{\nu}),$$

  where (x_Γ, ν) is the unique solutin of $x = x_\Gamma + \nu n(x_\Gamma)$ and $(\hat{x}, \hat{\nu}) := \big(\psi_\Gamma(x_\Gamma), \nu\big)/\delta$.
- χ_η is defined for $x \in C_{\delta,\eta}$ by $\chi_\eta(x) := \chi\left(\frac{\nu}{\eta}\right)$ with (x_Γ, ν) is the unique solutin of $x = x_\Gamma + \nu n(x_\Gamma)$ if $x_\Gamma \in C_{\delta,\eta}$. This last map is extended by zero for $x \notin C_{\delta,\eta}$.
- Here χ is a smooth cut off function such that $\chi \equiv 1$ on $]-\infty, 1[$ and $\chi \equiv 0$ on $]2, \infty[$.

We will estimate in this chapter for all $n \in \mathbb{N}$ the error $E^\delta - E^n_{\eta,\delta}$.

10.1 Stability of the exact problem

For all open set $\tilde{\Omega} \subset \mathbb{R}^3$, we define:

$$H(\mathbf{rot}; \tilde{\Omega}) := \left\{ u \in L^2(\tilde{\Omega}^\delta)^3,\ \mathbf{rot}(u) \in L^2(\tilde{\Omega}^\delta)^3 \right\},$$

and the norm of this space is given for $u \in H(\mathbf{rot}; \tilde{\Omega})$ by:

$$\|u\|^2_{H(\mathbf{rot};\tilde{\Omega})} := \|u\|^2_{L^2(\tilde{\Omega}^\delta)^3} + \|\mathbf{rot}(u)\|^2_{L^2(\tilde{\Omega}^\delta)^3}.$$

We introduce the spaces: $L^2_{\mathrm{comp}}(\Omega^\delta) := \Big\{ f \in L^2(\Omega^\delta)^3,\ \mathrm{supp}(f) \text{ is compact} \Big\}$ and

$$V^\delta := \left\{ (E^\delta, H^\delta) \in L^2_{\mathrm{loc}}(\Omega^\delta)^2,\ \begin{pmatrix} \mathbf{rot} E^\delta + ik\mu^\delta H^\delta, \\ \mathbf{rot} H^\delta - ik\epsilon^\delta E^\delta \end{pmatrix} \in L^2_{\mathrm{comp}}(\Omega^\delta)^2,\ E^\delta \times n = 0 \text{ on } \Gamma_\delta \text{ satisfaying (8.1.3)} \right\}$$

Next we define introduce the operator $\mathbb{P}^\delta : V^\delta \mapsto L^2_{\mathrm{comp}}(\Omega^\delta)^2$ defined for $(E^\delta, H^\delta) \in V^\delta$ by:

$$\mathbb{P}^\delta \begin{pmatrix} E^\delta \\ H^\delta \end{pmatrix} = \begin{pmatrix} \mathbf{rot} E^\delta + ik\mu^\delta H^\delta \\ \mathbf{rot} H^\delta - ik\epsilon^\delta E^\delta \end{pmatrix}.$$

Thanks to these notations, we can rewrite our exact problem in the following form: Find $\big(E^\delta, H^\delta\big) \in V^\delta$ such that:

$$\mathbb{P}_\delta \begin{pmatrix} E^\delta \\ H^\delta \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Thanks to the classical maxwell theory of time harmonics scattering (see [58]) we easily get existence of the inverse of this last operator. However we need a uniform continuity property with the small parameter $\delta > 0$:

Lemma 10.1.1. *For all $R > 0$ with $\Omega \subset B_R$ there exist $C_R > 0$ independent of $\delta > 0$ such that for all $f \in L^2_{\mathrm{comp}}(\Omega^\delta)^2$ we have:*

$$\mathrm{supp}(f) \subset B_R \cap \Omega^\delta \Longrightarrow \left\| \mathbb{P}_\delta^{-1} \cdot f \right\|_{H(\mathbf{rot}, B_R \cap \Omega^\delta)^2} \leq C \|f\|_{L^2(B_R \cap \Omega^\delta)},$$

where B_R is the open ball of radius R centered at zero.

The proof of this result is made by contradiction. We have chosen to split the proof in several propositions in order to to clarify its main steps.

If this result is false then: There exist $R > 0$ and a sequence $(E^\delta, H^\delta) \in V^\delta$ such that we have:

$$\lim_{\delta \to 0} \left\| \mathbb{P}_\delta (E^\delta, H^\delta) \right\|_{L^2(B_R \cap \Omega^\delta)} = 0 \quad \text{and} \quad \left\| E^\delta \right\|_{H(\mathbf{rot}, B_R \cap \Omega^\delta)} + \left\| H^\delta \right\|_{H(\mathbf{rot}, B_R \cap \Omega^\delta)} = 1, \tag{10.1.1}$$

and we will prove during whole this section that this last proposition bring a contradiction.

First let us prove that we can reduce to the fixed domain Ω. We can easily prove (with the local coordinate transformation $\mathcal{L}$) the existence of a sequence of transformation $\mathbb{I}_\delta : \Omega \mapsto \Omega_\delta$ such that:

$$\exists C > 0, \forall \delta > 0,\ \|\mathbb{I}_\delta\|_{C^1(\Omega)} + \|\mathbb{I}_\delta^{-1}\|_{C^1(\Omega^\delta)} \leq C, \tag{10.1.2}$$

and

$$\forall x \in \Omega,\ \mathrm{dist}(x, \Gamma) > \delta \Longrightarrow \mathbb{I}_\delta(x) = x. \tag{10.1.3}$$

From this last transformation we define the two vectors field:

$$E^\delta_\star := \mathrm{D}^\dagger\, \mathbb{I}_\delta \cdot \Big(E^\delta \circ \mathbb{I}_\delta \Big) \quad \text{and} \quad H^\delta_\star := \mathrm{D}^\dagger\, \mathbb{I}_\delta \cdot \Big(H^\delta \circ \mathbb{I}_\delta \Big),$$

the tensor field of coefficients:

$$\epsilon^\delta_\star := \big(\epsilon^\delta \circ \mathbb{I}_\delta\big) \det(\mathrm{D}\, \mathbb{I}_\delta)\, \mathrm{D}\, \mathbb{I}_\delta^{-1} \cdot \mathrm{D}\, \mathbb{I}_\delta^{-\dagger} \quad \text{and} \quad \mu^\delta_\star := \big(\mu^\delta \circ \mathbb{I}_\delta\big) \det(\mathrm{D}\, \mathbb{I}_\delta)\, \mathrm{D}\, \mathbb{I}_\delta^{-1} \cdot \mathrm{D}\, \mathbb{I}_\delta^{-\dagger},$$

and the vector field of right-hand side:

$$f^\delta_{\star,E} := \mathbf{rot} E^\delta_\star + ik\mu^\delta_\star H^\delta_\star \quad \text{and} \quad f^\delta_{\star,H} := \mathbf{rot} H^\delta_\star - ik\epsilon^\delta_\star E^\delta_\star. \tag{10.1.4}$$

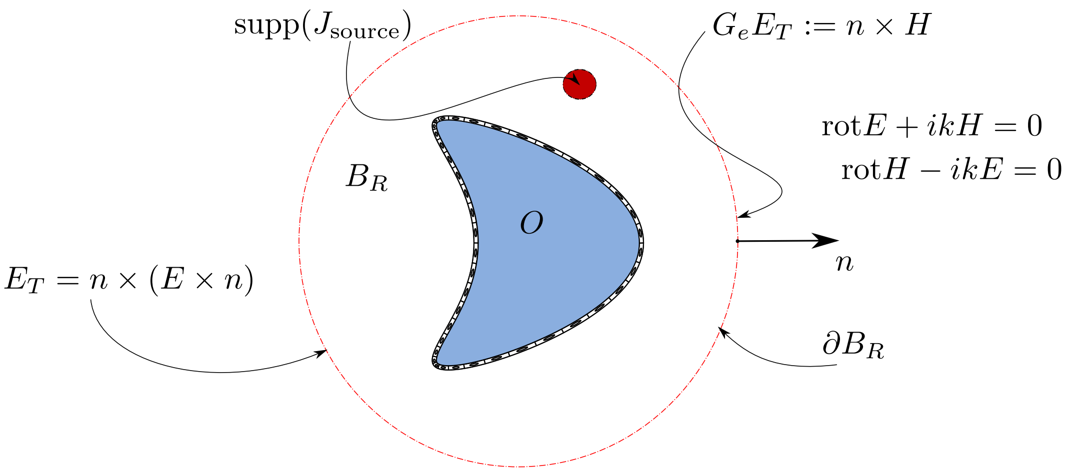


Figure 10.1: Illustration of the Calderon map

Proposition 10.1.2. *There exists $C > 0$ such that:*

$$C^{-1} \leq \left\|E_\star^\delta\right\|_{H(\mathbf{rot},\Omega)} + \left\|H_\star^\delta\right\|_{H(\mathbf{rot},\Omega)} \leq C. \tag{10.1.5}$$

and we have:

$$\lim_{\delta\to 0} f_{\star,E}^\delta = 0 \quad \textit{and} \quad \lim_{\delta\to 0} f_{\star,H}^\delta = 0 \textit{ in } L^2(B_R \cap \Omega), \tag{10.1.6}$$

Proof. It is a direct consequence of change variable formula for rotational operator(see [58, Corollary 3.58]), (10.1.1) and (10.1.2). □

Since our domain Ω is not bounded, we introduce the Calderon operator

$$G_e : H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R) \mapsto H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R),$$

(see [58, 9.4 Electromagnetic Calderon operators]) where:

$$H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R) := \left\{u \in H^{-\frac{1}{2}}\left(\partial B_R\right)^3, u \text{ is tangential and } \mathrm{div}_{\partial B_R}(u) \in H^{-\frac{1}{2}}\left(\partial B_R\right)\right\}.$$

We introduced this operator to reduce our analysis in the bounded domain $B_R \cap \Omega$. We recall that this last operator is defined for $E_T \in H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R)$ by $G_e u_T := n \times H$ where E, H is the unique solution of: Find $(E, H) \in H_{\mathrm{loc}}(\mathbf{rot}, \Omega)^2$ such that we have:

$$\mathbf{rot}E + ikH = 0 \quad \text{and} \quad \mathbf{rot}H - ikE = 0 \text{ with } E \times n = E_T \text{ on } \Gamma \text{ satisfying (8.1.3)}.$$

Moreover we introduce the following space:

$$X := \left\{u \in H(\mathbf{rot}, \Omega \cap B_R), u \times n = 0 \text{ on } \Gamma\right\},$$

equipped with the norm of $H(\mathbf{rot}, \Omega \cap B_R)$.

Proposition 10.1.3. *The following proposition:*

$$\lim_{\delta\to 0} E_\star^\delta = 0 \textit{ in } X \tag{10.1.7}$$

contradicts the proposition (10.1.5).

Proof. It is a direct consequence of (10.1.4) and Proposition 10.1.2 $\square$

Hereafter $\langle\cdot,\cdot\rangle$ is the dual bracket $\langle\cdot,\cdot\rangle_{H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R};\partial B_R)-H^{-\frac{1}{2}}(\mathrm{rot}_{\partial B_R};\partial B_R)}$ where:

$$H^{-\frac{1}{2}}(\mathrm{rot}_{\partial B_R};\partial B_R) := \left\{u \in H^{-\frac{1}{2}}\left(\partial B_R\right)^3, u \text{ is tangential and } \mathrm{rot}_{\partial B_R}(u) \in H^{-\frac{1}{2}}\left(\partial B_R\right)\right\},$$

and $(\cdot,\cdot)$ is the classical dot product on $L^2(\partial B_R)$. The normal unit u is extended to ∂B_R. Then we can introduce the sesquilinear form $a^\delta : X \times X \mapsto \mathbb{C}$ defined for $(u,v) \in X \times X$ by:

$$a^\delta_\star(u,v) = \Big((\mu^\delta_\star)^{-1}\mathbf{rot}u, \mathbf{rot}v\Big) - k^2\big(\epsilon^\delta_\star u, v\big) + ik\big\langle G_e(n \times u), (n \times v) \times n\big\rangle.$$

Proposition 10.1.4. *We have:*

$$\lim_{\delta\to 0}\sup_{\phi\in X}\frac{a^\delta_\star(E^\delta_\star,\phi)}{\|\phi\|_X} = 0. \tag{10.1.8}$$

Proof. We have for all $\phi \in X$ that:

$$a^\delta_\star(E^\delta_\star,\phi) = (l^\delta,\phi),$$

where we defined the anti-linear form $l^\delta \in X^\dagger$ for $\phi \in X$ by:

$$(l^\delta,\phi) = -ik\int_{B_R\cap\Omega} f^\delta_{\star,H}\overline{\phi}d\Omega + k^2\int_{B_R\cap\Omega}\big((\mu^\delta_\star)^{-1}f^\delta_{\star,E},\mathbf{rot}(\phi)\big)d\Omega.$$

Thanks to (10.1.6) we easily get $\lim_{\delta\to 0} l^\delta = 0$ in $X^\dagger$ which conclude the proof. $\square$

Nevertheless the sesquilinear form is not coercive and we can not now deduce the convergence (10.1.7). However we have the following result:

Proposition 10.1.5. *The following proposition:*

$$\lim_{\delta\to 0} E^\delta_\star = 0 \textit{ weakly in } X \quad \textit{and} \quad \lim_{\delta\to 0} E^\delta_\star = 0 \textit{ in } L^2(B_R\cap\Omega^\delta),$$

is a sufficient condition to the proposition (10.1.7).

Proof. From [58, Lemma 10.5,Theorem 10.6] we get that the operator G_e takes the following decomposition:

$$G_e = G^1_e + G^2_e,$$

where $ikG^1_e : H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R};\partial B_R) \mapsto H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R};\partial B_R)$ is a positive operator and G^2_e is an operator such that the following sesquilinear form defined on $X \times X$:

$$(u,v) \mapsto \big\langle G^2_e(n\times u),(n\times v)\times n\big\rangle,$$

is compact. Therefore the sesquilinear form $a^\delta_\star$ takes the following decomposition:

$$a^\delta_\star = c^\delta_\star + k^\delta_\star,$$

where we defined for $(u,v) \in X \times X$ the sesquilinear form:

$$\begin{cases} c^\delta_\star(u,v) := \displaystyle\int_{B_R\cap\Omega}\Big((\mu^\delta_\star)^{-1}\mathbf{rot}u,\mathbf{rot}v\Big) + \big(\epsilon^\delta_\star u,v\big)d\Omega + ik\big\langle G^1_e(n\times u),(n\times v)\times n\big\rangle, \\ k^\delta_\star(u,v) := \displaystyle\int_{B_R\cap\Omega} -(1+k^2)\big(\epsilon^\delta_\star u,v\big)d\Omega + ik\big\langle G^2_e(n\times u),(n\times v)\times n\big\rangle. \end{cases}$$

The positivity property of the operator G_e^1 and the assumption (8.2.4) yields that a the sesquilinear form $c_\star^\delta$ is uniformly coercive with the small parameter. Therefore a sufficient condition for (10.1.7) is:

$$\lim_{\delta\to 0} c_\star^\delta(E_\star^\delta, E_\star^\delta) = 0. \tag{10.1.9}$$

Let us prove now Since $E_\star^\delta$ is bounded on X then (10.1.8) leads to $\lim\limits_{\delta\to 0} a_\star^\delta(E_\star^\delta, E_\star^\delta) = 0$. Now let us prove that:

$$\lim_{\delta\to 0} k_\star^\delta(E_\star^\delta, E_\star^\delta) = 0. \tag{10.1.10}$$

On the one hand, thanks to $\lim\limits_{\delta\to 0} E_\star^\delta$ in $L^2(B_R \cap \Omega)$ we have:

$$\lim_{\delta\to 0} \int_{B_R\cap\Omega} -(1+k^2)\big(\epsilon_\star^\delta E_\star^\delta, E_\star^\delta\big) d\Omega = 0.$$

On the other hand, thanks to $\lim\limits_{\delta\to 0} E_\star^\delta$ weakly in the space X and the compactness property of the operator G_e^2 we have:

$$\lim_{\delta\to 0} ik\big\langle G_e^2(n\times E_\star^\delta), (n\times E_\star^\delta)\times n\big\rangle = 0,$$

which concludes the proof of (10.1.10). Combining this with (10.1.10) and (10.1.8) concludes the proof of (10.1.9) which ends the proof. □

Proposition 10.1.6. *We have the following convergence:*

$$\lim_{\delta\to 0} E_\star^\delta = 0 \textit{ weakly in } X$$

Proof. Let C be the subspace of X constituted of the element $u \in X$ such that $u \equiv 0$ in the neighborhood of Γ. Let $\phi \in C$. Thanks to (10.1.3), we have for δ small enough we have:

$$a_\star^\delta(E_\star^\delta, \phi) = a_\star^0(E_\star^\delta, \phi).$$

Combining this with (10.1.8) yields that

$$\lim_{\delta\to 0} a_\star^0(E_\star^\delta, \phi) = 0. \tag{10.1.11}$$

Since the sequence $(E_\star^\delta)_{\delta>0}$ is bounded in the space X then there exits $E_\star^0 \in X$ such that $(E_\star^\delta)_{\delta>0}$ weakly converge up to a subsequence to $E_\star^0 \in X$. Using the continuity of the sesquilinear form $a_\star^0$ and the weak convergence yields:

$$\lim_{\delta\to 0} a_\star^0(E_\star^\delta, \phi) = a_\star^0(E_\star^0, \phi). \tag{10.1.12}$$

Combining this last convergence with (10.1.11) yields:

$$a_\star^0(E_\star^0, \phi) = 0, \tag{10.1.13}$$

and using that C is dense in the space X yields that this last equality remains valid for all $\phi \in X$. Thanks to [58, Reduction to a bounded domain] we get that (10.1.13) implies that $E_\star^0 = 0$ which concludes the proof. □

Thanks to [58, Theorem 10.2] we can introduce the potential $\phi^\delta_\star$ defined by the unique solution of: Find $\phi^\delta_\star \in S$ such that for all $\psi \in S$ we have:

$$a^\delta_\star(\nabla\phi^\delta_\star, \nabla\psi) = a^\delta_\star(E^\delta_\star, \nabla\psi), \tag{10.1.14}$$

where we defined the following closed subspace of $H^1(B_R \cap \Omega)$:

$$S := \left\{ u \in H^1(B_R \cap \Omega), u = 0 \text{ on } \Gamma \right\}.$$

We emphasize that we have $\nabla S \subset X$.

Proposition 10.1.7. *We have the following convergence:*

$$\lim_{\delta \to 0} \phi^\delta_\star = 0 \ \textit{in } S$$

Proof. Assume that the result is false. Then the sequence $(\psi^\delta_\star)_{\delta>0}$ defined for δ by:

$$\psi^\delta_\star := \frac{\psi^\delta_\star}{\|\psi^\delta_\star\|_S},$$

is bounded and weakly converge up to a subsequence to some $\psi^0_\star \in S$. We define the sesquilinear form $a^\delta_\nabla : S \times S \mapsto \mathbb{C}$ for $(u, v) \in S \times S$ by:

$$a^\delta_\nabla(u, v) := a^\delta_\star(\nabla u, \nabla v),$$

and thanks to (10.1.8) we get:

$$\lim_{\delta \to 0} \sup_{\phi \in S} \frac{a^\delta_\nabla(\psi^\delta_\star, \phi)}{\|\phi\|_S} = 0. \tag{10.1.15}$$

Let us prove that $\psi^0_\star = 0$. Indeed for all ϕ smooth function vanishing in the neighborhood of Γ we have for small δ that $a^\delta_\nabla(\psi^0_\star, \phi) = a^0_\nabla(\psi^0_\star, \phi)$. Combining with the continuity of the sesquilinear form a^0_∇ , the weak convergence and (10.1.15) that:

$$a^0_\nabla(\psi^0_\star, \phi) = \lim_{\delta \to 0} a^\delta_\nabla(\psi^\delta_\star, \phi) = 0.$$

Using an argument of density yields that this last equality remains true for all $\phi \in S$ and thanks to [58, Theorem 10.2] we get that $\psi^0_\star = 0$.

Let us prove the existence of a coercive (with a constant of coercivity independent of $\delta > 0$) sesquilinear form c^δ_∇ and compact sesquilinear form k^δ_∇ such that the following decomposition holds:

$$a^\delta_\nabla = -c^\delta_\nabla + k_\nabla. \tag{10.1.16}$$

Indeed, thanks to [58, Lemma 9.23, Theorem 10.2] there exists an operator $\tilde{G}_e : H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R) \mapsto H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R)$ such that the sesquilinear form k^δ_∇ defined for $(u, v) \in S$:

$$k_\nabla(u, v) := ik \left\langle (G_e + ik\tilde{G}_e)(n \times \nabla u), n \times (\nabla v \times n) \right\rangle.$$

is compact. Moreover [58, Lemma 9.23, Theorem 10.2] also states that for all $u \in S$ we have:

$$k^2 \left\langle \tilde{G}_e(n \times \nabla u), n \times (\nabla u \times n) \right\rangle \leq 0.$$

Combining this with the assumption (8.2.4), yields that the sesquilinear form c^δ_∇ defined for $(u,v)\in S^2$ by:

$$c^\delta_\nabla(u,v) := k^2\big(\epsilon^\delta_\star \nabla u, \nabla v\big) - k^2\left\langle \tilde{G}_e(n\times\nabla u), n\times(\nabla u\times n)\right\rangle,$$

is well uniformly coercive with the parameter δ which concludes the proof.

Since we have proved $\psi^0_\star = 0$, the compactness of the sesquilinear form k_∇ leads to:

$$\lim_{\delta\to 0} k_\nabla(\psi^\delta_\star,\psi^\delta_\star) = 0.$$

Combining this last convergence with (10.1.15) and the decomposition (10.1.16) yields:

$$\lim_{\delta\to 0} c^\delta_\nabla(\psi^\delta_\star,\psi^\delta_\star) = 0.$$

Using the uniform coercivity property of the sesquilinear form c^δ_∇ yields that:

$$\lim_{\delta\to 0} \psi^\delta_\star = 0 \text{ in } S,$$

which is absurd. □

Proposition 10.1.8. *Up to a subsequence we have that the sequence $(E^\delta_\star - \nabla\phi^\delta_\star)_{\delta>0}$ converge in the space $L^2(B_R\cap\Omega)$.*

Proof. We will use the existing compactness properties found in [35, Lemma 2.3] and [58, Lemma 10.4] of the two following space:

$$\begin{cases} X^0_\delta := \left\{u\in H_0(\mathbf{rot},\Omega\cap B_R), \operatorname{div}(\epsilon^\delta_\star u) = 0\right\}, \\ X^1_0 := \left\{u\in X, \forall\psi\in S,\ a^0_\star(u,\nabla\psi) = 0\right\}, \end{cases}$$

that we recall here:

- For all bounded sequence $(U_\delta)_{\delta>0}$ in the space X if for all $\delta>0$ we have $U_\delta\in X^0_\delta$ then there exists a subsequence of $\delta>0$ such that $(U_\delta)_{\delta>0}$ converge in the space $L^2(B_R\cap\Omega)$.
- The space X^1_0 is compactly embedded in the space $L^2(B_R\cap\Omega)$.

Nevertheless the vector $E^\delta_\star - \nabla\phi^\delta_\star$ neither belongs to the space X^0_δ or the space X^1_0. However we succeed to prove that this last vector take the following forms:

$$E^\delta_\star - \nabla\phi^\delta_\star = \big(v^\delta_0 - \nabla\phi^\delta_{\star,0}\big) + \big(v^\delta_1 - \nabla\phi^\delta_{\star,1}\big) + \nabla\big(\phi^\delta_{\star,0}+\phi^\delta_{\star,1}\big), \tag{10.1.17}$$

where we now prove that:

1. The function $\phi^\delta_{\star,0}$ and $\phi^\delta_{\star,1}$ converge up to a subsequence in the space S.
2. The sequence $(v^\delta_0 - \nabla\phi^\delta_{\star,0})_{\delta>0}$ is bounded in X and belongs to X^0_δ for all δ.
3. $(v^\delta_1 - \nabla\phi^\delta_{\star,1})_{\delta>0}$ is bounded in X and belongs to X^1_0 for all δ.

The vector field v_0^δ and v_1^δ are defined by

$$v_0^\delta := \chi \cdot (E_\star^\delta - \phi_\star^\delta) \quad \text{and} \quad v_1^\delta := (1-\chi)\cdot(E_\star^\delta - \phi_\star^\delta).$$

where χ is a C^∞ cut off function function such that $\chi \equiv 1$ in the neighborhood of Γ and $\chi \equiv 0$ in the neighborhood of ∂B_R and the functions $(\phi_{\star,0}^\delta, \phi_{\star,1}^\delta) \in H_0^1(B_R \cap \Omega) \times S$ are defined by the unique solution of the two following problems:

- Find $\phi_{\star,0}^\delta \in H_0^1(B_R \cap \Omega)$ such that for all $\psi \in H_0^1(B_R \cap \Omega)$ we have:

$$(\epsilon_\star^\delta \nabla \phi_{\star,0}^\delta, \nabla \psi) = \big(l_{\star,0}^\delta, \psi\big).$$

- Find $\phi_{\star,1}^\delta \in S$ such that for all $\psi \in S$ we have:

$$a_\star^0(\nabla \phi_{\star,1}^\delta, \nabla\psi) = (l_{\star,1}^\delta, \psi).$$

Here the anti-linear forms $(l_{\star,0}^\delta, l_{\star,1}^\delta) \in H_0^1(B_R\cap\Omega)^\dagger \times S^\dagger$ for $(\psi^0,\psi^1) \in H_0^1(B_R \cap \Omega^\delta)\times S$ by:

$$\big(l_{\star,0}^\delta, \psi\big) := (\epsilon_\star^\delta v_0^\delta, \nabla\psi) \quad \text{and} \quad \big(l_{\star,1}^\delta, \psi\big) := a_\star^0(v_1^\delta, \nabla\psi).$$

Now let us prove the proposition 1.

Thanks to the hypothesis (8.2.4) and [58, Theorem 10.2] a sufficient condition is to prove that the sequence $(l_{\star,0}^\delta)_{\delta>0}$ and $(l_{\star,1}^\delta)_{\delta>0}$ converge up to a subsequence in the spaces $H_0^1(B_R\cap\Omega)^\dagger$ and $S^\dagger$. According to the Rellich lemma, a sufficient condition is to prove that these two sequences of anti-linear forms are bounded in the space $L^2(B_R\cap\Omega)^\dagger$. Indeed thanks to (10.1.14), we have for all $\psi \in S$ that:

$$a_\star^\delta\big(E_\star^\delta - \nabla\phi_\star^\delta, \nabla(\chi\psi)\big) = 0 \quad \text{and} \quad a_\star^0\Big(E_\star^\delta - \nabla\phi_\star^\delta, \nabla\big((1-\chi)\psi\big)\Big) = 0. \tag{10.1.18}$$

Therefore for all $\psi \in H_0^1(B_R \cap \Omega^\delta)$ we have

$$\begin{aligned}\big(l_{\star,0}^\delta, \psi\big) = (\epsilon_\star^\delta \chi\cdot(E_\star^\delta - \nabla\phi_\star^\delta), \nabla\psi) &= (\epsilon_\star^\delta (E_\star^\delta - \nabla\phi_\star^\delta), \chi\cdot\nabla\psi),\\ &= (\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta), \nabla(\chi\psi)) - (\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta)\cdot\nabla\chi, \psi)\\ &= \cancel{a_\star^\delta((E_\star^\delta - \nabla\phi_\star^\delta), \nabla(\chi\psi))} - (\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta)\cdot\nabla\chi, \psi),\end{aligned}$$

which leads to $l_{\star,0}^\delta = \epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta)\cdot\nabla\chi$ in $\big(H_0^1(B_R\cap\Omega)\big)^\dagger$. Moreover the sequence $(E_\star^\delta - \nabla\phi_\star^\delta)_{\delta>0}$ is bounded in $L^2(B_R\cap\Omega)$ which conclude the proof for $l_{\star,0}^\delta$. Thanks to (10.1.18) we have for all $\psi\in S$:

$$\begin{aligned}\big(l_{\star,1}^\delta, \psi\big) =& \Big(\epsilon_\star^\delta(1-\chi)\cdot(E_\star^\delta - \nabla\phi_\star^\delta), \nabla\psi\Big) + \Big\langle G_e((1-\chi)(E_\star^\delta - \nabla\phi_\star^\delta), n\times(\nabla\psi\times n)\Big\rangle,\\ =& \Big(\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta), (1-\chi)\nabla\psi\Big) + \Big\langle G_e((E_\star^\delta - \nabla\phi_\star^\delta), n\times(\nabla((1-\chi)\psi)\times n)\Big\rangle,\\ =& \Big(\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta), \nabla\big((1-\chi)\psi\big)\Big) + \Big\langle G_e((E_\star^\delta - \nabla\phi_\star^\delta), n\times(\nabla((1-\chi)\psi)\times n)\Big\rangle +\\ & \Big(\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta)\cdot\nabla(1-\chi), \psi\Big),\\ =& \cancel{a_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta, (1-\chi)\psi)} + \Big(\epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta)\cdot\nabla(1-\chi), \psi\Big)\end{aligned}$$

which leads to $l_{\star,1}^\delta = \epsilon_\star^\delta(E_\star^\delta - \nabla\phi_\star^\delta)\cdot\nabla(1-\chi)$ in $S^\dagger$ and conclude the proof for $l_{\star,1}^\delta$.

The proposition 2 and 3 are direct consequence of the definition of the function $\phi_{\star,0}^\delta$ and $\phi_{\star,1}^\delta$ and then we have finished our proof. $\square$

Proof of Lemma 10.1.1. Combining Proposition 10.1.7 with Proposition 10.1.8 yields that the sequence $(E^\delta_\star)_{\delta>0}$ converges in the space $L^2(B_R \cap \Omega)$. Therefore thanks to Proposition 10.1.6 we get:

$$\lim_{\delta\to 0} E^\delta_\star = 0 \text{ in } L^2(B_R \cap \Omega),$$

and using Proposition 10.1.5 yields that (10.1.7) is fulfilled. Therefore thanks to Proposition 10.1.3 we get that (10.1.5) is false which bring a contradiction. Therefore we finish our proof. □

10.2 Error decomposition

Thanks to Lemma 10.1.1 it remains to estimate the following quantities:

$$\begin{cases} Q^{E,n}_{\eta,\delta} := \left\|\mathbf{rot}(E^\delta - E^n_{\eta,\delta}) + ik\mu^\delta(H^\delta - H^n_{\eta,\delta})\right\|_{L^2(\Omega^\delta)}, \\ Q^{H,n}_{\eta,\delta} := \left\|\mathbf{rot}(H^\delta - H^n_{\eta,\delta}) - ik\epsilon^\delta(E^\delta - E^n_{\eta,\delta})\right\|_{L^2(\Omega^\delta)}. \end{cases}$$

We only give the proof of estimate for the quantity $Q^{E,n}_{\eta,\delta}$ because the one of $Q^{H,n}_{\eta,\delta}$ is exactly the same. We split this last quantity into the following form: $Q^{E,n}_{\eta,\delta}$:

$$Q^{E,n}_{\eta,\delta} = \mathcal{D}^c_{\eta,\delta,n} + \mathcal{D}^r_{\eta,\delta,n},$$

where $\mathcal{D}^c_{\eta,\delta,n}$ is so-called "consistency error" (it measures how much the truncated expansion (9.1.6) fails to satisfy the original Maxwell equation):

$$\mathcal{D}^c_{\eta,\delta,n} := \left\|\mathbf{rot}\big(\mathcal{I}_{\delta,\eta}\left(\hat{E}^n_\delta\right)\big) + ik\mu^\delta\,\mathcal{I}_{\delta,\eta}(\hat{H}^n_\delta)\right\|_{L^2(C_{\delta,\eta})},$$

where $\mathcal{D}^r_{\eta,\delta,n}$ is so-called "matching error" (it measures the mismatch between the truncated expansions (9.1.6) and (9.1.2)):

$$\mathcal{D}^r_{\eta,\delta,n} := \left\|\mathbf{rot}\big(\chi_\eta(\mathcal{I}_{\delta,\eta}\,\hat{E}^n_\delta - E^n_\delta)\big) + ik\mu^\delta\chi_\eta\big(\mathcal{I}_{\delta,\eta}\,\hat{H}^n_\delta - H^n_\delta\big)\right\|_{L^2\left(C_{\eta,2\eta}\right)}.$$

10.3 Estimate of the consistency error

We have the following decomposition of the consistency error:

$$\mathcal{D}^c_{\eta,\delta,n} \le C\cdot\big(\mathcal{D}^{c,0}_{\eta,\delta,n} + \mathcal{D}^{c,1}_{\eta,\delta,n}\big).$$

Here, we defined:

- The "first consistency error" by:

  $$\mathcal{D}^{c,0}_{\eta,\delta,n} := \left\|\mathbf{rot}\big(\mathcal{I}_{\delta,\eta}\,\hat{E}^n_\delta\big) + ik\mathcal{I}^n\mu^\delta\,\mathcal{I}_{\delta,\eta}\,\hat{H}^n_\delta\right\|_{L^2(C_{\delta,\eta})},$$

  where we defined for $x \in C_{\delta,\eta}$ the function $\mathcal{I}^n(x) := \mathcal{M}^{-1}(x_\Gamma,\nu)\sum_{j=0}^{n}\mathcal{M}_j(x_\Gamma)\nu^j$ where $(x_\Gamma,\nu) \in \Gamma\times]-\delta,\eta[$ is the unique solution of $x = x_\Gamma + \nu n(x_\Gamma)$.

- The "second consistency error" by:

  $$\mathcal{D}^{c,1}_{\eta,\delta,n} := \left\|ik(\mathcal{I}^n - 1)\mu^\delta\,\mathcal{I}_{\delta,\eta}\,\hat{H}^n_\delta\right\|_{L^2(C_{\delta,\eta})}.$$

10.3.1 Estimate of the first consistency error

We introduce the following quantity:

$$Q_n^\delta := -\delta^n \hat{\mathbf{rot}}\left(\hat{E}^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k},$$

because we have the following result:

Proposition 10.3.1. *One has for all $x \in C_{\delta,\eta}$:*

$$\left(\mathbf{rot}\left(\mathcal{I}_{\delta,\eta}\, \hat{E}_\delta^n\right) + ik\mathcal{I}^n \mu^\delta\, \mathcal{I}_\delta\, \hat{H}_\delta^n\right)(x) = \mathcal{M}(x_\Gamma, \nu)^{-1}\left(\mathcal{I}_{\delta,\eta}\, Q_n^\delta\right)(x),$$

where $(x_\Gamma, \nu) \in \Gamma \times]-\delta, \eta[$ is the unique solution of $x = x_\Gamma + \nu n(x_\Gamma)$.

Proof. Thanks to Proposition 9.2.3, we have:

$$\left(\mathbf{rot}\left(\mathcal{I}_{\delta,\eta}\, \hat{E}_\delta^n\right) + ik\mathcal{I}^n \mu^\delta\, \mathcal{I}_\delta\, \hat{H}_\delta^n\right)(x) = \mathcal{M}(x_\Gamma, \nu)^{-1}\left(\mathcal{I}_{\delta,\eta}\, \tilde{Q}_n^\delta\right)(x),$$

where we defined:

$$\tilde{Q}_n^\delta := \delta^{-1} \hat{\mathbf{rot}}\left(\hat{E}_\delta^n\right) + \mathbf{rot}_\Gamma\left(\hat{E}_\delta^n\right) + ik\mu \sum_{j=0}^{n} \mathcal{M}_j \hat{\nu}^j \delta^j \hat{H}_\delta^n.$$

Therefore it remains to prove that $\tilde{Q}_n^\delta = Q_n^\delta$. Indeed we have:

$$\begin{aligned}
\tilde{Q}_n^\delta =& \delta^{-1} \hat{\mathbf{rot}}\left(\hat{E}_\delta^n\right) + \mathbf{rot}_\Gamma\left(\hat{E}_\delta^n\right) + ik\mu \sum_{j=0}^{n} \mathcal{M}_j \hat{\nu}^j \delta^j \hat{H}_\delta^n, \\
=& \sum_{j=0}^{n} \delta^{j-1} \hat{\mathbf{rot}}\left(\hat{E}^j\right) + \sum_{j=0}^{n} \delta^j \mathbf{rot}_\Gamma\left(\hat{E}^j\right) + ik\mu \sum_{j=0}^{n} \mathcal{M}_j \hat{\nu}^j \sum_{k=0}^{n} \delta^{k+j} \hat{H}^k, \\
=& \sum_{j=-1}^{n-1} \delta^j \hat{\mathbf{rot}}\left(\hat{E}^{j+1}\right) + \sum_{j=0}^{n} \delta^j \mathbf{rot}_\Gamma\left(\hat{E}^j\right) + ik\mu \sum_{(k,j)\in N_1} \mathcal{M}_j \hat{\nu}^j \delta^{j+k} \hat{H}^k,
\end{aligned}$$

where $N_1 := \{(k,l) \in \mathbb{Z}^2,\ 0 \le k \le n \quad \text{and} \quad 0 \le l \le n\}$. Let $\mathcal{N} : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$ defined for $(k,l) \in \mathbb{Z}^2$ by:

$$\mathcal{N}(k,l) := (k, l+k),$$

and we remark that this application is bijective. From the following equivalence:

$$\forall (k,l) \in \mathbb{Z}^2, \quad \begin{cases} 0 \le k \le n \\ 0 \le l \le n \end{cases} \Longleftrightarrow \left(\begin{cases} 0 \le k \le l+k \\ 0 \le l+k \le n \end{cases} \text{or} \quad \begin{cases} l+k-n \le k \le n \\ n+1 \le l+k \le 2n \end{cases}\right),$$

we get that $\mathcal{N}(N_1) = N_2^1 \cup N_2^2$ with $N_2^1 \cap N_2^2 = \emptyset$ and:

$$\begin{cases} N_1 := \left\{(k,l) \in \mathbb{Z}^2,\ 0 \le k \le l \quad \text{and} \quad 0 \le l \le n\right\}, \\ N_2 := \left\{(k,l) \in \mathbb{Z}^2,\ l-n \le k \le n \quad \text{and} \quad n+1 \le l \le 2n\right\}. \end{cases}$$

$$\sum_{(k,j)\in N_1} \mathcal{M}_j \hat{\nu}^j \delta^{j+k} \hat{H}^k = \sum_{(k,j)\in \mathcal{N}(N_1)} \delta^l \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k} \qquad = \sum_{m=1,2} \sum_{(k,j)\in N_2^m} \delta^j \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k}.$$

Moreover combining with (9.2.18) which is stated Lemma 9.3.11 yields:

$$\begin{aligned}
\tilde{Q}_n^\delta &= \sum_{j=-1}^{n-1} \delta^j \hat{\mathbf{rot}} \left(\hat{E}^{j+1}\right) + \sum_{j=0}^{n} \delta^j \mathbf{rot}_\Gamma \left(\hat{E}^j\right) + ik\mu \sum_{m=1,2} \sum_{(k,j)\in N_2^m} \delta^j \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k}, \\
a &= \cancel{\sum_{j=0}^{n} \delta^j \hat{\mathbf{rot}} \left(\hat{E}^{j+1}\right) + \sum_{j=0}^{n} \delta^j \mathbf{rot}_\Gamma \left(\hat{E}^j\right)} - \delta^n \hat{\mathbf{rot}} \left(\hat{E}^{n+1}\right) \\
&\quad + \cancel{ik\mu \sum_{j=0}^{n} \delta^j \sum_{k=0}^{j} \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k}} + ik\mu \sum_{(k,j)\in N_2^2} \delta^j \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k}, \\
&= -\delta^n \hat{\mathbf{rot}} \left(\hat{E}^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k} = Q_\delta^n,
\end{aligned}$$

which conclude the proof. □

Proposition 10.3.2. *For all $m \in \mathbb{N}$ the following estimate holds:*

$$\|Q_\delta^n\|_{C^m(\Gamma; L^2(]0,1[^2\times]-1,\eta/\delta[))} \leq C\eta^{n+\frac{1}{2}}\delta^{-\frac{1}{2}}$$

Proof. We recall that:

$$Q_\delta^n = -\delta^n \hat{\mathbf{rot}} \left(\hat{E}^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k \hat{H}^{j-k},$$

and thanks to the decomposition (9.3.65) found in Lemma 9.3.11 we have:

$$\begin{aligned}
Q_\delta^n &= -\, \delta^n \hat{\mathbf{rot}} \left(R_E^{n+1} + P_E^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k (R_H^{j-k} + P_H^{j-k}), \\
&= -\, \delta^n \hat{\mathbf{rot}} \left(R_E^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k (R_H^{j-k}) - \\
&\quad \delta^n \partial_{\hat{\nu}} \left(n \times P_E^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k (P_H^{j-k}),
\end{aligned}$$

which leads to the following decomposition:

$$Q_\delta^n := \delta^n Q_\delta^{n,0} + Q_\delta^{n,1}, \tag{10.3.19}$$

where we defined:

$$\begin{cases}
Q_\delta^{n,0} := -\hat{\mathbf{rot}} \left(R_E^{n+1}\right) + ik\mu \sum_{j=n+1}^{2n} \delta^{j-n} \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k (R_H^{j-k}), \\
Q_\delta^{n,1} := \delta^n \cdot n \times \partial_{\hat{\nu}} P_E^{n+1} + ik\mu \sum_{j=n+1}^{2n} \delta^j \sum_{k=l-j}^{n} \mathcal{M}_k \hat{\nu}^k (P_H^{j-k}).
\end{cases}$$

We can easily prove that for all m the quantity $Q_\delta^{n,0}$ satisfies the $\mathcal{P}_m^\infty$ property which leads to:

$$\left\|Q_\delta^{n,0}\right\|_{C^m(\Gamma;L^2(]0,1[^2\times]-1,\eta/\delta[))} \leq C. \tag{10.3.20}$$

Moreover we have existence of $P_A \in C^\infty(\Gamma;\mathbb{C}_n[\hat{\nu}])$ and $(P_B^{n+1},\cdots,P_B^{2n}) \in C^\infty(\Gamma;\mathbb{C}_{n+1}[\hat{\nu}])\times\cdots\times C^\infty(\Gamma;\mathbb{C}_{2n}[\hat{\nu}])$ such that we have:

$$Q_\delta^{n,1} = \delta^n P_A + ik\mu \sum_{j=n+1}^{2n} \delta^j P_B^j.$$

In the scalar case we have already shown that for all $p \in \mathbb{N}$ and $P \in C^\infty(\Gamma;\mathbb{C}_p[\hat{\nu}])$ we have:

$$\|\mathrm{D}_\Gamma^m P\|_{C^m(\Gamma;L^2(]0,1[^2\times]-1,\eta/\delta[))} \leq C\left(\frac{\eta}{\delta}\right)^{p+\frac{1}{2}}, \tag{10.3.21}$$

which to leads the following estimate:

$$\left\|Q_\delta^{n,1}\right\|_{C^m(\Gamma;L^2(]0,1[^2\times]-1,\eta/\delta[))} \leq C\left(\delta^n\left(\frac{\eta}{\delta}\right)^{n+\frac{1}{2}} + \sum_{j=n+1}^{2n}\delta^j\left(\frac{\eta}{\delta}\right)^{j+\frac{1}{2}}\right) \leq C\eta^{n+\frac{1}{2}}\delta^{-\frac{1}{2}}.$$

Combining this last estimate with (10.3.19) and (10.3.20) conclude our proof. □

Corollary 10.3.3. *The first consistency error satisfies the following estimate:*

$$\mathcal{D}_{\eta,\delta,n}^{c,0} \leq C\eta^{n+\frac{1}{2}}.$$

Proof. It is a direct consequence of Proposition 10.3.2 and Proposition 3.1.1 (See Chapter 3), □

10.3.2 Estimate of the second consistency error

Proposition 10.3.4. *We have for all $m \in \mathbb{N}$ the following estimate:*

$$\left\|\hat{H}_\delta^n\right\|_{C^m(\Gamma;L^2(]0,1[^2\times]-1,\eta/\delta[))} \leq C\eta^{\frac{1}{2}}\delta^{-\frac{1}{2}}.$$

Proof. It is a direct consequence of (9.3.65) found in Lemma 9.3.11 and the estimate (10.3.21). □

Corollary 10.3.5. *The first consistency error satisfies the following estimate:*

$$\mathcal{D}_{\eta,\delta,n}^{c,1} \leq C\eta^{n+\frac{3}{2}}.$$

Proof. Since $(x_\Gamma,\nu) \mapsto \mathcal{M}^n(x_\Gamma,\nu)$ is the Taylor expansion of order n in ν of the function M^n, we have that the function:

$$x \mapsto \frac{\mathcal{I}(x)-1}{\nu^{n+1}},$$

where $(x_\Gamma,\nu) \in \Gamma\times]-\delta,\eta[$ is the unique solution of $x = x_\Gamma + n(x_\Gamma)\nu$ is bounded. Moreover the function μ^δ is bounded then we have existence of $C>0$ independent of δ such that:

$$\mathcal{D}_{\eta,\delta,n}^{c,1} = \left\|ik(\mathcal{I}^n-1)\mu^\delta\,\mathcal{I}_\delta(\hat{H}_\delta^n)\right\|_{L^2(\Gamma\times]-\delta,\eta[)} \leq C\eta^{n+1}\left\|\mathcal{I}_\delta(\hat{H}_\delta^n)\right\|_{L^2(\Gamma\times]-\delta,\eta[)}.$$

Combining this last estimate with Proposition 3.1.1 (See Chapter 3), and Proposition 10.3.4 yields:

$$\mathcal{D}^{c,1}_{\eta,\delta,n} \leq C\eta^{n+1}\delta^{\frac{1}{2}} \left\| \hat{H}^n_\delta \right\|_{C^3(\Gamma;L^2(]0,1[^2\times]-1,\eta/\delta[))} \leq C\eta^{n+\frac{3}{2}},$$

which conclude the proof. □

10.3.3 Total estimate of the consistency error

Thanks to Corollary 10.3.3 an Corollary 10.3.5 we get the following result:

Lemma 10.3.6. *There exists $C > 0$ independent of δ such that the following estimate holds:*

$$\mathcal{D}^c_{\eta,\delta,n} \leq C\eta^{n+\frac{1}{2}}$$

10.4 Estimate of matching error

Lemma 10.4.1. *There exists $C > 0$ independent of δ such that the followings estimate holds:*

$$\mathcal{D}^r_{\eta,\delta,n} \leq C\Big(\eta^{n+\frac{1}{2}} + \delta^{-1}\exp\Big(-\pi g_{\min}\frac{\eta}{\delta}\Big)\Big).$$

Proof. We skip the proof as it is similar to the one of Lemma 3.3.4 (See Chapter 3). We recall that this result is a consequence of the C^∞ regularity of the far field and the identities (9.1.10). In this case, proceeding as previously a δ^{-1} term appears, intrinsically due to Maxwell equation equation(oscillating term already appears in the term $\hat{E}_0$ contrary to the term $\hat{u}_0 \equiv u_0(\nu = 0)$. □

10.5 Justification and error estimate theorem

Theorem 10.5.1. *For all bounded open set $K \subset \Omega$ satisfying $\overline{K} \cap \Gamma = \emptyset$ there exists $C_K > 0$ such that the following estimate holds:*

$$\left\| E^\delta - E^n_{\delta,\eta} \right\|_{H(\mathbf{rot},K)} + \| H_\delta - H^n_\delta \|_{H(\mathbf{rot},K)} \leq C\delta^{n+1}.$$

Proof. First we emphasize that we can chose the function $\delta \mapsto \eta(\delta) := \delta^{\frac{n}{n+\frac{1}{2}}}$ as it last one well satisfies the required property that we recall above:

$$\lim_{\delta\to 0}\eta(\delta) = 0 \quad \text{and} \quad \lim_{\delta\to 0}\frac{\eta(\delta)}{\delta} = \infty.$$

Then we prove that for this specific choice we have existence of $C > 0$ such that for all bounded open set K the following estimate holds.

$$\left\| E^\delta - E^n_{\delta,\eta} \right\|_{H(\mathbf{rot},K)} + \left\| H_\delta - H^n_{\delta,\eta} \right\|_{H(\mathbf{rot},K)} \leq C\delta^n. \qquad (10.5.22)$$

Let $K \subset \Omega$ be a bounded open set. Thanks to the result of consistence error estimate Lemma 10.4.1, the result of matching error estimate Lemma 10.3.6 and the stability result Lemma 10.1.1 we first get the following estimate:

$$\left\| E^\delta - E^n_{\delta,\eta} \right\|_{H(\mathbf{rot},K)} + \left\| H_\delta - H^n_{\delta,\eta} \right\|_{H(\mathbf{rot},K)} \leq C\Big(\eta^{n+\frac{1}{2}} + \delta^{-1}\exp\Big(-\pi g_{\min}\frac{\eta}{\delta}\Big)\Big). \qquad (10.5.23)$$

Thanks to these last property we get existence of $C > 0$ independent of $\delta > 0$ such that:

$$\exp\left(-\pi g_{\min}\frac{\eta}{\delta}\right) \leq C\left(\frac{\delta}{\eta}\right)^{(n+1)^2} = C\delta^{n+1},$$

and combining this last estimate with (10.5.23) conclude the proof of the estimate (10.5.22). Now that we proved this estimate we now use the hypothesis $\overline{K}\cap\Gamma$. In this case it is easy to prove that for δ small enough that we have $\chi_\eta \equiv 0$ and then the estimate (10.5.22) becomes:

$$\left\|E^\delta - E_\delta^n\right\|_{H(\mathbf{rot},K)} + \|H_\delta - H_\delta^n\|_{H(\mathbf{rot},K)} \leq C\delta^n.$$

Since the proof is true for all n then this last one is also true for $n+1$ which yields:

$$\left\|E^\delta - E_\delta^{n+1}\right\|_{H(\mathbf{rot},K)} + \left\|H_\delta - H_\delta^{n+1}\right\|_{H(\mathbf{rot},K)} \leq C\delta^{n+1}.$$

Therefore we have:

$$\begin{aligned}\left\|E^\delta - E_\delta^n\right\|_{H(\mathbf{rot},K)} + \|H_\delta - H_\delta^n\|_{H(\mathbf{rot},K)} \leq \left\|E^\delta - E_\delta^{n+1}\right\|_{H(\mathbf{rot},K)} + \left\|H_\delta - H_\delta^{n+1}\right\|_{H(\mathbf{rot},K)} + \\ \delta^{n+1}\left\|H^{n+1}\right\|_{H(\mathbf{rot},K)} + \delta^{n+1}\left\|E^{n+1}\right\|_{H(\mathbf{rot},K)} \leq C\delta^{n+1},\end{aligned}$$

which is the stated estimate in this theorem. $\square$

Chapter 11

Effective boundary condition of order 1

11.1 Explicit construction of the far field and the near field for $n = 0$

11.1.1 The far field

Applying the formula (9.3.62), yields that the far field (E^0, H^0) are defined by the unique solution of: Find $(E^0, H^0) \in H_{\mathrm{loc}}(\mathbf{rot}; \Omega)^2$ such that:

$$\mathbf{rot} E^0 = -ikH^0 \qquad \text{and} \qquad \mathbf{rot} H^0 = ikE^0 + J_{\mathrm{source}},$$

satisfying the boundary condition $\gamma_t E^0 = 0$ on Γ and the radiating condition (8.1.3).

11.1.2 The near field

Applying (9.3.63) and (9.3.64) with $n = 0$ yields for all $(x_\Gamma; \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$:

$$\hat{E}^0(x_\Gamma; \hat{x}, \hat{\nu}) = (E^0(x_\Gamma) \cdot n(x_\Gamma)) \mathcal{N}_E(x_\Gamma; \hat{x}, \hat{\nu}), \tag{11.1.1}$$

and

$$H^0(x_\Gamma; \hat{x}, \hat{\nu}) = \mathcal{N}_H(x_\Gamma; \hat{x}, \hat{\nu}) \Big(n(x_\Gamma) \times \big(H^0(x_\Gamma) \times n(x_\Gamma) \big) \Big). \tag{11.1.2}$$

11.2 Explicit construction of the far field for $n = 1$

We introduce for convenience the normal trace operator for $u : \Omega \mapsto \mathbb{C}^3$ and $x_\Gamma \in \Gamma$ by:

$$\gamma_n u(x_\Gamma) := u(x_\Gamma) \cdot n(x_\Gamma).$$

Thanks to (11.1.1) and (11.1.2), we have for $n = 1$ and all $(x_\Gamma, \hat{x}, \hat{\nu}) \in \Gamma \times \hat{\Omega}$ that :

$$f_1^E(x_\Gamma; \hat{x}, \hat{\nu}) = -\mathbf{rot}_\Gamma \Big(\big(E^0(x_\Gamma) \cdot n(x_\Gamma) \big) \mathcal{N}_E(x_\Gamma; \hat{x}, \hat{\nu}) \Big) \tag{11.2.3}$$

$$- ik\hat{\mu}(x_\Gamma; \hat{x}, \hat{\nu}) \mathcal{N}_H(x_\Gamma; \hat{x}, \hat{\nu}) \Big(n(x_\Gamma) \times \big(H^0(x_\Gamma) \times n(x_\Gamma) \big) \Big). \tag{11.2.4}$$

Applying (9.3.62) yields that the far field (E^1, H^1) are defined by the unique solution of: Find $(E^1, H^1) \in H_{\text{loc}}(\mathbf{rot}; \Omega)^2$ such that:

$$\mathbf{rot} E^1 = -ikH^1 \qquad \text{and} \qquad \mathbf{rot} H^1 = ikE^1 \text{ in } \Omega,$$

satisfying the boundary condition:

$$E^1(x_\Gamma) \times n(x_\Gamma) = \int_{\hat{Y}_-} f_1^E(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu},$$

and the radiating condition (8.1.3). To express more explicitly this last quantity we introduce the scalar field $\boldsymbol{\epsilon}_{\text{eff}}^{-1}$ defined for $x_\Gamma \in \Gamma$ by:

$$\boldsymbol{\epsilon}_{\text{eff}}^{-1}(x_\Gamma) := \int_{\hat{Y}_\infty} \hat{\epsilon}(x_\Gamma) \left| \widehat{\nabla}\, w^\epsilon(x_\Gamma; \hat{x}, \hat{\nu}) \right|^2 d\hat{x} d\hat{\nu},$$

and the tensor field $\boldsymbol{\mu}_{\text{eff}}$ defined for $x_\Gamma \in \Gamma$ by:

- If $x_\Gamma \in \Gamma_{\text{M}}$ then $\boldsymbol{\mu}_{\text{eff}}(x_\Gamma)$ is the unique element of $\mathcal{L}\big(T_{x_\Gamma}\Gamma\big)$ such that for all $(i,j) \in \{1,2\}^2$ we have:

$$\big(\boldsymbol{\mu}_{\text{eff}}(x_\Gamma) e_i(x_\Gamma), e_j(x_\Gamma)\big) = \int_{\hat{Y}_\infty} \hat{\mu} \Big(\big(e_i(x_\Gamma), e_j(x_\Gamma)\big) + \big(\widehat{\nabla}\, w_i(x_\Gamma; \hat{x}, \hat{\nu}), \widehat{\nabla}\, w_j(x_\Gamma; \hat{x}, \hat{\nu}) \big) \Big) d\hat{x} d\hat{\nu}.$$

- If $x_\Gamma \notin \Gamma_{\text{M}}$ then:

$$\boldsymbol{\mu}_{\text{eff}}(x_\Gamma) := \int_{\hat{Y}_\infty} \hat{\mu}(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} \mathbb{I}.$$

Finally we define the operator $\mathcal{Z}_1$ for u by:

$$\mathcal{Z}_1 u = k^{-2} \vec{\mathbf{rot}}_\Gamma \big(\boldsymbol{\epsilon}_{\text{eff}}^{-1} \operatorname{rot}_\Gamma u \big) - \boldsymbol{\mu}_{\text{eff}} u.$$

The space of definition of this last operator will be given later.

Proposition 11.2.1. *For all $x_\Gamma \in \Gamma$, we have*

$$\int_{\hat{Y}_-} f_1^E(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = ik \mathcal{Z}_1(\gamma_T H^0)(x_\Gamma).$$

Proof. First we prove that we have:

$$\int_{\hat{Y}_-} \mathbf{rot}_\Gamma \Big(\gamma_n E^0 \mathcal{N}_E(\cdot; \hat{x}, \hat{\nu}) \Big) d\hat{x} d\hat{\nu} = -(ik)^{-1} \vec{\mathbf{rot}}_\Gamma \Big(\boldsymbol{\epsilon}_{\text{eff}}^{-1} \operatorname{rot}_\Gamma \big(\gamma_T H^0 \big) \Big). \tag{11.2.5}$$

To prove this last equality we need to prove that we have:

$$\forall x_\Gamma \in \Gamma, \int_{\hat{Y}_-} \mathcal{N}_E(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \boldsymbol{\epsilon}_{\text{eff}}^{-1}(x_\Gamma; \hat{x}, \hat{\nu}) n(x_\Gamma). \tag{11.2.6}$$

Indeed, let $x_\Gamma \in \Gamma$. Using the periodicity of the function w^ϵ and the definition of the vector $\mathcal{N}_E$ given in (9.3.26) yields:

$$\int_{\hat{Y}_-} \mathcal{N}_E(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = \int_{\hat{Y}_-} \widehat{\nabla}\, w^\epsilon(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu} = n(x_\Gamma) \cdot \int_{\hat{Y}_-} \partial_{\hat{\nu}} w^\epsilon(x_\Gamma; \hat{x}, \hat{\nu}) d\hat{x} d\hat{\nu}, \tag{11.2.7}$$

and using (9.3.25) with w^ϵ as a test function yields:

$$\int_\Sigma w^\epsilon(x_\Gamma;\hat{x},\hat{\nu})d\hat{x} = \int_{\hat{Y}_\infty}\hat{\epsilon}\big(\widehat{\nabla}\, w^\epsilon(x_\Gamma;\hat{x},\hat{\nu}),\widehat{\nabla}\, w^\epsilon(x_\Gamma;\hat{x},\hat{\nu})\big)d\hat{x}d\hat{\nu} = \boldsymbol{\epsilon}_{\text{eff}}^{-1}(x_\Gamma).$$

Therefore combining this last equality with (11.2.7) end the proof of (11.2.6). Thus using (11.2.6) yields:

$$\mathbf{rot}_\Gamma\left(\int_{\hat{Y}_-}\gamma_n E^0\mathcal{N}_E(\cdot;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\right) = \vec{\mathbf{rot}}_\Gamma\left(\gamma_n E^0\boldsymbol{\epsilon}_{\text{eff}}^{-1}\right).$$

Moreover thanks to the equation $\mathbf{rot}E^0 = -ikH^1$ we have $\gamma_n E^0 = -(ik)^{-1}\text{rot}_\Gamma\big(\gamma_T H^0\big)$.

Secondly we prove that for all x_Γ we have:

$$ik\int_{\hat{Y}_-}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu})H^0(x_\Gamma)d\hat{x}d\hat{\nu} = ik\boldsymbol{\mu}_{\text{eff}}\big(n(x_\Gamma)\times(H^0(x_\Gamma)\times n(x_\Gamma))\big). \quad (11.2.8)$$

Indeed, for $x_\Gamma\in\Gamma_{\text{M}}$ we have for all $i\in\{1,2\}$:

$$\int_{\hat{Y}_-}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu})e_i(x_\Gamma)d\hat{x}d\hat{\nu} = \hat{\boldsymbol{\mu}}_{\text{eff}}(x_\Gamma)e_i(x_\Gamma), \quad (11.2.9)$$

because applying (9.3.30) with $w_i(x_\Gamma;\cdot)$ as a test function yields:

$$\begin{aligned}\left(\int_{\hat{Y}_-}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\mathcal{N}_H(x_\Gamma;\hat{x},\hat{\nu})e_i(x_\Gamma)d\hat{x}d\hat{\nu},e_j(x_\Gamma)\right) &= \int_{\hat{Y}_-}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})\big(\widehat{\nabla}\, w_i(x_\Gamma;\hat{x},\hat{\nu}),\widehat{\nabla}\, w_j(x_\Gamma;\hat{x},\hat{\nu})\big)d\hat{x}d\hat{\nu},\\ &+\int_{\hat{Y}_-}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})d\hat{x}d\hat{\nu}\big(e_i(x_\Gamma),e_j(x_\Gamma)\big),\\ &=\big(\boldsymbol{\mu}_{\text{eff}}(x_\Gamma)e_i(x_\Gamma),e_j(x_\Gamma)\big).\end{aligned}$$

The case of $x_\Gamma\notin\Gamma$ is trivial Finally combining (11.2.5) and (11.2.8) with (11.2.3) ends the proof. □

11.3 Deduction to an effective boundary condition

11.3.1 Formal deduction

From the previous section we recall that

$$\gamma_t E^0 = 0 \quad\text{and}\quad \gamma_t E^1 = ik\mathcal{Z}_1\big(\gamma_T H^0\big),$$

which leads to:

$$\gamma_t\left(E^0+\delta E^1\right)\approx ik\delta\mathcal{Z}_1\big(\gamma_T(H^0+\delta H^1)\big). \quad (11.3.10)$$

More precisely we have

$$\gamma_t\left(E^0+\delta E^1\right) = \delta\mathcal{Z}_1\big(\gamma_T(H^0+\delta H^1)\big)+O(\delta^2)\quad\text{with}\quad O(\delta^2):=-ik\delta^2\mathcal{Z}_1\big(\gamma_T H^1\big).$$

Thus we introduce the field (E_1^δ,H_1^δ) as the unique solution of (8.1.1) and (8.1.3) satisfying the following boundary condition(the study of this problem will be done further):

$$\gamma_t E_1^\delta = ik\delta\mathcal{Z}_1\big(\gamma_T H_1^\delta\big).$$

11.3.2 Variational formulation

We introduce for $\delta > 0$ the following Hilbert space:

$$V^\delta := \left\{U \in H(\mathbf{rot}; B_R \cap \Omega), \mathrm{rot}_\Gamma(U) \in L^2(\Gamma)\right\},$$

and we provide this last with the norm defined for $U \in V^\delta$ by:

$$\|U\|_{V^\delta} := \delta^{\frac{1}{2}} \left(\|U\|_{L^2(\Gamma)} + \|\mathrm{rot}_\Gamma(U)\|_{L^2(\Gamma)}\right) + \|U\|_{H(\mathbf{rot}, B_R \cap \Omega)}.$$

We define on this last space the sesquilinear form a_1^δ defined for $(H, H') \in V^\delta$ by:

$$\begin{aligned} a_1^\delta(H, H') :=& \delta \int_\Gamma \left(\boldsymbol{\epsilon}_{\text{eff}}^{-1} \mathrm{rot}_\Gamma(H) \cdot \overline{\mathrm{rot}_\Gamma(H')} - k^2 \big(\boldsymbol{\mu}_{\text{eff}} H, H'\big)\right) \mathrm{d}\Gamma +, \\ & \int_{B_R \cap \Omega} \Big(\big(\mathbf{rot} H, \mathbf{rot} H'\big) - k^2 \big(H, H'\big)\Big) \mathrm{d}\Omega + ik \big\langle G_e(n \times H), (n \times H) \times n \big\rangle \end{aligned}$$

We introduce this last sesquilinear because we can easily prove that the restriction the function H_1^δ on the domain $B_R \cap \Omega$ is the unique solution of: Find $H_1^\delta \in V_\delta$ such that we have for all $H' \in V_\delta$ we have:

$$a_1^\delta(H_1^\delta, H') = \int_{B_R \cap \Omega} J_{\text{source}} \cdot \mathbf{rot}(H').$$

11.3.3 Consistency error

We need to introduce the two following space:

$$H^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma) := \left\{ u \in H^{-\frac{1}{2}}(\Gamma)^3, u \text{ is tangential and } \mathrm{div}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma) \right\},$$

and

$$H^{-\frac{1}{2}}(\mathrm{rot}_\Gamma; \Gamma) := \left\{ u \in H^{-\frac{1}{2}}(\Gamma)^3, u \text{ is tangential and } \mathrm{rot}_\Gamma u \in H^{-\frac{1}{2}}(\Gamma) \right\}.$$

We recall that:

$$H^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma) = \left(H^{-\frac{1}{2}}(\mathrm{rot}_\Gamma; \Gamma)\right)^\dagger \quad \text{and} \quad H^{-\frac{1}{2}}(\mathrm{rot}_\Gamma; \Gamma) = \left(H^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma)\right)^\dagger \qquad (11.3.11)$$

Lemma 11.3.1. *There exists $C > 0$ independent of δ such that we have the following consistency error:*

$$\sup_{H' \in V^\delta} \frac{a_1^\delta(H^0 + \delta H^1 - H_1^\delta, H')}{\|H'\|_{V^\delta}} \leq C\delta^2$$

Proof. Let $H' \in V^\delta$ satisfying $\|H'\|_{V^\delta} = 1$. We will show an uniform estimate to the small parameter δ and H' of the quantity $a_1^\delta(H^0 + \delta H^1 - H_1^\delta, H')$. Indeed thanks to (11.3.10) and (11.3.11) we have:

$$\begin{aligned} a_1^\delta(H^0 + \delta H^1 - H_1^\delta, H') &= \delta^2 \left(l_\Gamma, \gamma_T H'\right)_{L^2(\Gamma)}, \\ &\leq \delta^2 \left\|l_\Gamma\right\|_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma)} \cdot \left\|\gamma_T H'\right\|_{H^{-\frac{1}{2}}(\mathrm{rot}_\Gamma; \Gamma)}, \\ &\leq C\delta^2 \left\|l_\Gamma\right\|_{H^{-\frac{1}{2}}(\mathrm{div}_\Gamma; \Gamma)} \|H'\|_{H(\mathbf{rot}; \Omega)}, \end{aligned}$$

where we posed the quantity $l_\Gamma := \vec{\mathbf{rot}}_\Gamma\big(\boldsymbol{\epsilon}_{\text{eff}}^{-1}\text{rot}_\Gamma(\gamma_T u)\big) - k^2 \boldsymbol{\mu}_{\text{eff}}\gamma_T u$. Therefore combining this with $\|H'\|_{H(\mathbf{rot};\Omega)} \leq \|H'\|_{V^\delta} = 1$ leads to :

$$a_1^\delta(H^0 + \delta H^1 - H_1^\delta, H') \leq C\|l_\Gamma\|_{H^{-\frac{1}{2}}(\mathbf{rot};\Gamma)}.$$

Therefore it remains to prove that $l_\Gamma \in H^{-\frac{1}{2}}(\text{div}_\Gamma;\Gamma)$. This result is a direct consequence of the regularity of the far field H^1 and the effective coefficients $\boldsymbol{\epsilon}_{\text{eff}}^{-1}$ and $\boldsymbol{\mu}_{\text{eff}}$. □

11.4 Stability of the effective boundary condition

Lemma 11.4.1. *There exists $C > 0$ independent of δ such that we have for all $H \in V^\delta$:*

$$\|H\|_{V^\delta} \leq C \sup_{H' \in V^\delta} \frac{a_1^\delta(H, H')}{\|H'\|_{V^\delta}}.$$

As the proof of Lemma 10.1.1, we will prove this lemma by contradiction. If this result is false then there exists a sequence $H^\delta \in V^\delta$ such that we have:

$$\lim_{\delta \to 0} \sup_{H' \in V^\delta} \frac{a_1^\delta(H^\delta, H')}{\|H'\|_{V^\delta}} = 0 \quad \text{and} \quad \left\|H^\delta\right\|_{V^\delta} = 1. \tag{11.4.12}$$

Nevertheless, we do not have compactness result of the injection from V^δ into $L^2(\Omega) \cap L^2(\Gamma)$. However, we have a compactness result stated Proposition 11.4.3 for the following space:

$$X_0^\delta := \left\{H \in V^\delta, a_1^\delta(H, \nabla\psi) = 0, \ \forall \psi \in S^\delta\right\},$$

where we defined the following space:

$$S^\delta := H^1(B_R \cap \Omega) \cap H^1(\Gamma).$$

We emphasize that from the relation $\text{rot}_\Gamma \nabla_\Gamma = 0$ that we have the following inclusion:

$$\nabla S^\delta \subset V^\delta.$$

Although we do not have $H^\delta \in X_0^\delta$, we can reduce to this last case by using the the following result:

Proposition 11.4.2. *There exists $C > 0$ independent of $\delta > 0$ such that the sesquilinear form a_1^δ satisfies the following* $\inf - \sup$ *condition:*

$$\inf_{\psi \in S^\delta} \sup_{\psi' \in S^\delta} \frac{a_1^\delta\Big(\nabla\phi, \nabla\psi'\Big)}{\|H\|_{S^\delta}\|H'\|_{S^\delta}} \geq C \quad \textit{and} \quad \inf_{\psi \in S^\delta} \sup_{\psi' \in S^\delta} \frac{a_1^\delta\Big(\nabla\phi', \nabla\psi\Big)}{\|H\|_{S^\delta}\|H'\|_{S^\delta}} \geq C.$$

Proof. The proof is exactly the same than the one of the stability result for the Helmholtz equation stated in Lemma 4.5.4 (See Chapter 4). It is sufficient to replace the Dirichlet to Neumann map by the following sesquilinear forms defined for $(u, v) \in H^1(B_R \cap \Omega)^2$ by:

$$ik\langle G_e \cdot (n \times \nabla u), n \times (\nabla v \times n)\rangle.$$

Indeed this last sesquilinear satisfies the same required properties than DtN we used to do the proof of Lemma 4.5.4 (See Chapter 4). □

Indeed thanks to this last result we have existence and uniqueness of a sequence $\psi^\delta \in S^\delta$ such that we have:

$$a_1^\delta(\nabla\psi^\delta, \nabla\psi') = -a_1^\delta(H^\delta, \nabla\psi'), \tag{11.4.13}$$

satisfying the the following estimate:

$$\|\psi^\delta\|_{S^\delta} \leq C^{-1} \sup_{\psi' \in S^\delta} \frac{a_1^\delta(H^\delta, \nabla\psi')}{\|\psi'\|_{S^\delta}}, \tag{11.4.14}$$

where $C > 0$ is the constant appearing in Proposition 11.4.2. Combining the estimate (11.4.14) with (11.4.12) yields that:

$$\lim_{\delta\to 0} \|\nabla\psi^\delta\|_{V^\delta} = 0,$$

and thanks to (11.4.13) we directly get that $H^\delta + \nabla\psi^\delta \in X_0^\delta$.

Proposition 11.4.3. *Let $(H^\delta)_{\delta>0}$ be a sequence of element of X_0^δ such that there exists $C > 0$ independent of δ with:*

$$\|H^\delta\|_{V^\delta} \leq C, \tag{11.4.15}$$

Then there exists $H \in L^2(B_R \cap \Omega)$ and $H_\Gamma \in L^2(\Gamma)$ such that we have up to a sub-sequence the following convergence:

$$\lim_{\delta\to 0} H^\delta = H \text{ in } L^2(\Omega) \quad \text{and} \quad \lim_{\delta\to 0} \delta^{\frac{1}{2}} H^\delta = H_\Gamma \text{ in } L^2(\Gamma). \tag{11.4.16}$$

Moreover there exists g_Γ such that up to a sub-sequence we have:

$$\lim_{\delta\to 0} G_e^2(n \times H^\delta) = g_\Gamma. \tag{11.4.17}$$

The proof is inspired from the one of [38, Lemma 15]. We introduce the half space $\mathbb{P} := \mathbb{R}^2\times]-\infty, 0[$. We denote for vector $u : \mathbb{P} \mapsto \mathbb{C}^3$ the tangential trace by $u_{\partial\mathbb{P}}$. Moreover $\mathrm{div}_{\partial\mathbb{P}}$ and $\mathbf{rot}_{\partial\mathbb{P}}$ are the surface divergence and scalar curl on $\partial\mathbb{P}$. We need also to define the following assumptions for $(M, P) \in C^1\big(\partial\mathbb{P}; \mathcal{M}_2(\mathbb{R})\big) \times C^1\big(\mathbb{P}; \mathcal{M}_3(\mathbb{R})\big)$, $\eta > 0$ and $u \in L^2(\mathbb{P})^3$:

(I) There exists $C_{M,P}, C' > 0$ such that for all $v \in H^1(\mathbb{P})^3$ with $\mathrm{div}(v) = 0$ and $v_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^2$ we have:

$$\begin{aligned} -\mathrm{Re} \int_{\partial\mathbb{P}} \mathrm{div}_{\partial\mathbb{P}}(M v_{\partial\mathbb{P}}) \overline{\mathrm{div}_{\partial\mathbb{P}}(v_{\partial\mathbb{P}})} d(\partial\mathbb{P}) \leq C_{M,P} \|(Pv)_{\partial\mathbb{P}}\|^2_{H\left(\mathbf{rot}_{\partial\mathbb{P}}, \partial\mathbb{P}\right)} + \|Pv\|_{H(\mathbf{rot},\mathbb{P})} \\ - C'\left(\|v\|^2_{H^1(\mathbb{P})^3} + \|v_{\partial\mathbb{P}}\|^2_{H^1(\partial\mathbb{P})^2}\right). \end{aligned}$$

(II) We have the estimate : $\|P - \mathbb{I}\|_{C^1\left(\mathbb{P};\mathcal{M}_3(\mathbb{R})\right)} \leq \eta$.

(III) The matrix field M is uniformly definite positive. That means the existence of $C > 0$ such that: for all $(v, x) \in \mathbb{C}^2 \times \partial\mathbb{P}$ we have

$$\frac{(M(x)v, v)}{|v|^2} > C.$$

(IV) For all $x \in \partial\mathbb{P}$ and $v \in \mathbb{R}^2 \times \{0\}$: $P(x)v \in \mathbb{R}^2 \times \{0\}$.

(V) $\big(\mathbf{rot}(Pu), \mathbf{rot}_{\partial\mathbb{P}}(Pu)_{\partial\mathbb{P}}\big) \in L^2\big(\mathbb{P}\big)^3 \times L^2\big(\partial\mathbb{P}\big)$ and $(u, u_{\partial\mathbb{P}}) \in L^2(\mathbb{P})^3 \times L^2(\partial\mathbb{P})^2$.

(VI) For all $\phi : \Gamma \mapsto \mathbb{R}$ with $(\nabla\phi, \nabla_{\partial\mathbb{P}}\phi) \in L^2\big(\mathbb{P}\big)^3 \times L^2\big(\partial\mathbb{P}\big)^2$ we have:

$$\int_{\partial\mathbb{P}} Mu \cdot \nabla_{\partial\mathbb{P}}\phi d(\partial\mathbb{P}) + \int_{\mathbb{P}} u \cdot \nabla\phi d\mathbb{P} = 0.$$

To prove Proposition 11.4.3, one need the three following results:
Lemma 11.4.4. *Let* $(M, P) \in C^1\big(\partial\mathbb{P}; \mathcal{M}_2(\mathbb{R})\big) \times C^1\big(\mathbb{P}; \mathcal{M}_3(\mathbb{R})\big)$ *and* $\eta > 0$ *satisfying (I), (II), (III) and (IV). Then if* η *is small enough, we have for all* $u \in L^2(\mathbb{P})^3$ *satisfying (V), (VI) that* $u \in H^1(\mathbb{P})^3$ *and the following estimate holds:*

$$\|u\|^2_{H^1(\mathbb{P})} \leq 2\|Pu\|^2_{H(\mathbf{rot},\mathbb{P})} + 4C_{M,P}\|(Pu)_{\partial\mathbb{P}}\|^2_{H(\mathbf{rot}_{\partial\mathbb{P}},\partial\mathbb{P})}. \tag{11.4.18}$$

We will also use the following result which is found in [38, Lemma 14].
Lemma 11.4.5. *Let* M *be a definite positive matrix of size* 2×2 *and* $P = \mathbb{I}_3$*. Then (I), (III) and (IV) holds in this case.*
Lemma 11.4.6. *The set of matrix field* $(M, P) \in C^1\big(\partial\mathbb{P}; \mathcal{M}_2(\mathbb{R})\big) \times C^1\big(\mathbb{P}; \mathcal{M}_3(\mathbb{R})\big)$ *satisfying (I) and (III) is an open subset of* $C^1\big(\partial\mathbb{P}; \mathcal{M}_2(\mathbb{R})\big) \times C^1\big(\mathbb{P}; \mathcal{M}_3(\mathbb{R})\big)$*.*
Proof of Lemma 11.4.4. Let us prove first that $u \in H^1(\mathbb{P})^3$ and $(u)_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^3$ implies the estimate (11.4.18). In this case we can apply [60, Lemma 5.4.2], which leads to:

$$\|\mathbf{rot}(u)\|^2_{L^2(\mathbb{P})} + \|\mathrm{div}(u)\|^2_{L^2(\mathbb{P})} = \|\nabla u\|^2_{L^2(\mathbb{P})} + 2\mathrm{Re}\langle \mathrm{div}_\Gamma(u_\Gamma), u \cdot n\rangle_{\partial\mathbb{P}}, \tag{11.4.19}$$

where $\langle , \rangle_{\partial\mathbb{P}} := \langle \cdot, \cdot \rangle_{H^{-\frac{1}{2}}(\partial\mathbb{P})-H^{\frac{1}{2}}(\partial\mathbb{P})}$. Thanks to (II) if η is small enough then:

$$\|\mathbf{rot}(Pu)\|^2_{L^2(\mathbb{P})} \geq \|\mathbf{rot}(u)\|^2_{L^2(\mathbb{P})} - \frac{1}{2}\|u\|^2_{H^1(\mathbb{P})}.$$

Combining this last estimate with (11.4.19) leads to:

$$\|u\|^2_{L^2(\mathbb{P})} + \|\mathbf{rot}(Pu)\|^2_{L^2(\mathbb{P})} + \|\mathrm{div}(u)\|^2_{L^2(\mathbb{P})} \geq \frac{1}{2}\|u\|^2_{H^1(\mathbb{P})} + 2\mathrm{Re}\langle \mathrm{div}_\Gamma(u_\Gamma), u \cdot n\rangle_{\partial\mathbb{P}}. \tag{11.4.20}$$

The assumption (VI) leads to:

$$\mathrm{div}(u) = 0 \text{ in } \mathbb{P} \quad \text{and} \quad u \cdot n = \mathrm{div}_\Gamma(Mu_\Gamma) \text{ in } \partial\mathbb{P}. \tag{11.4.21}$$

Hence (11.4.20) becomes:

$$\|u\|^2_{L^2(\mathbb{P})} + \|\mathbf{rot}(Pu)\|^2_{L^2(\mathbb{P})} \geq \frac{1}{2}\|u\|^2_{H^1(\mathbb{P})} + 2\mathrm{Re}\langle \mathrm{div}_\Gamma(u_\Gamma), \mathrm{div}_\Gamma(Mu_\Gamma)\rangle_{\partial\mathbb{P}}, \tag{11.4.22}$$

Thanks to the assumption (I), this becomes:

$$\frac{1}{2}\|u\|^2_{H^1(\mathbb{P})} \leq \|u\|^2_{L^2(\mathbb{P})} + \|\mathbf{rot}(Pu)\|^2_{L^2(\mathbb{P})} + 2C_{M,P}\|(Pu)_{\partial\mathbb{P}}\|^2_{H\big(\mathbf{rot}_{\partial\mathbb{P}},\partial\mathbb{P}\big)},$$

which concludes the proof of the implication: $u \in H^1(\mathbb{P})$ and $(u)_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^3 \Rightarrow$ (11.4.18).

Therefore it remains to prove $u \in H^1(\mathbb{P})$ and $(u)_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^3$. Indeed thanks to (V), there exists a sequence $(u_j)_j$ with $u_j \in H^1(\mathbb{P})$ and $(u_j)_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^3$ such that:

$$Pu = \lim_{j\to\infty} Pu_j \text{ in } H(\mathbf{rot};\mathbb{P}) \quad \text{and} \quad (Pu)_{\partial\mathbb{P}} = \lim_{j\to\infty}(u_j)_{\partial\mathbb{P}} \text{ in } H(\mathbf{rot}_{\partial\mathbb{P}};\partial\mathbb{P}). \qquad (11.4.23)$$

Nevertheless this sequence a priory does not satisfies the assumption (VI). That is why we introduce for j a function ψ_j of the space:

$$V(\partial\mathbb{P}) := \left\{\psi \in L^2_{\text{lox}}(\mathbb{P})/\mathbb{C}, \|u\|^2_{V(\partial\mathbb{P})} := \int_{\partial\mathbb{P}} |\nabla u|^2 d\mathbb{P} + \int_{\mathbb{P}} |\nabla_{\partial\mathbb{P}} u|^2 d(\partial\mathbb{P}) < \infty\right\}.$$

This function is the unique solution of: Find $\psi_j \in V(\partial\mathbb{P})$ such that for all $\phi \in V(\partial\mathbb{P})$ we have:

$$a(\psi_j,\phi) = l_j(\phi). \qquad (11.4.24)$$

In this variational formulation, the sesquilinear form a and the anti-linear form l_j are defined for $(v,v') \in V(\partial\mathbb{P})^2$ by:

$$a(v,v') := \int_{\partial\mathbb{P}} (MP^{-1})\nabla_{\partial\mathbb{P}} v \cdot \nabla_{\partial\mathbb{P}} v' d(\partial\mathbb{P}) + \int_{\mathbb{P}} (P^{-1}\nabla v)\cdot \nabla v' d\mathbb{P},$$

and:

$$l_j(v') = -\int_{\partial\mathbb{P}} Mu_j \cdot \nabla_{\partial\mathbb{P}} v' d(\partial\mathbb{P}) - \int_{\mathbb{P}} u_j \cdot \nabla v' d\mathbb{P} = 0.$$

This last problem is well posed when η is small enough. Indeed, on the one hand we assumed that M is uniformly coercive on $\partial\mathbb{P}$. Combining this with (II) yields that if η is small enough then the matrix field MP^{-1} is also uniformly coercive on $\partial\mathbb{P}$. Moreover P^{-1} is also uniformly coercive on $\mathbb{P}$. Then the sesquilinear a form is well coercive in $V(\partial\mathbb{P})$. On the other hand the anti-linear form l_j is clearly continuous.

Thus we now can introduce the field:

$$\tilde{u}_j := u_j + P^{-1}\nabla\psi_j. \qquad (11.4.25)$$

Thanks to (11.4.24), this field well satisfies (VI). Thanks to the regularity of u_j and M, by inspiring from the proof of [57, Theorem 4.21], we can prove that: $\nabla\phi_j \in H^1(\mathbb{P})$ and $(\nabla\phi_j)_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^3$.

Therefore for all j,k, $\tilde{u}_j - \tilde{u}_k \in H^1(\mathbb{P})^3$ and $(\tilde{u}_j - \tilde{u}_k)_{\partial\mathbb{P}} \in H^1(\partial\mathbb{P})^2$. Moreover thanks to the properties $\mathbf{rot}\nabla = 0$ and $\mathbf{rot}_{\partial\mathbb{P}}\nabla_{\partial\mathbb{P}} = 0$ we have by construction:

$$\mathbf{rot}\big(P(\tilde{u}_j - \tilde{u}_k)\big) = \mathbf{rot}\big(P(u_j - u_k)\big) \in L^2(\mathbb{P})^3, \qquad (11.4.26)$$

and

$$\mathbf{rot}_{\partial\mathbb{P}}\big(P(\tilde{u}_j - \tilde{u}_k)\big) = \mathbf{rot}_{\partial\mathbb{P}}\big(P(u_j - u_k)\big)_{\partial\mathbb{P}} \in L^2(\mathbb{P}). \qquad (11.4.27)$$

Moreover, by linearity $\tilde{u}_j - \tilde{u}_k$ satisfies (V) and (VI). Thus we can apply the estimate (11.4.18) which leads to:

$$\|\tilde{u}_j - \tilde{u}_k\|^2_{H^1(\mathbb{P})} \le 2\|P(\tilde{u}_j - \tilde{u}_k)\|^2_{H(\mathbf{rot},\mathbb{P})} + 4C_{M,P}\big\|\big(P(\tilde{u}_j - \tilde{u}_k)\big)_{\partial\mathbb{P}}\big\|^2_{H(\mathbf{rot}_{\partial\mathbb{P}},\partial\mathbb{P})}.$$

Thanks to (11.4.26) and (11.4.27) this last estimate becomes:

$$\|\tilde{u}_j - \tilde{u}_k\|^2_{H^1(\mathbb{P})} \leq E^1_{jk} + E^2_{jk}, \tag{11.4.28}$$

where we defined for j, k:

$$E^1_{jk} := 2\|\mathbf{rot}\big(P(u_j - u_k)\big)\|^2_{L^2(\mathbb{P})^3} + 4C_{M,P}\|\mathbf{rot}_{\partial\mathbb{P}}\big(P(u_j - u_k)\big)_{\partial\mathbb{P}}\|^2_{L^2(\partial\mathbb{P})},$$

and

$$E^2_{jk} := 2\|(\tilde{u}_j - \tilde{u}_k)\|^2_{L^2(\mathbb{P})^3} + 4C_{M,P}\|(\tilde{u}_j - \tilde{u}_k)_{\partial\mathbb{P}}\|^2_{L^2(\partial\mathbb{P})}.$$

Now let us prove that:

$$\lim_{j\to\infty}\sup_{k\to\infty} E^1_{jk} = \lim_{j\to\infty}\sup_{k\to\infty} E^2_{jk} = 0. \tag{11.4.29}$$

Indeed, thanks to (11.4.23) we directly get that (because all convergent sequence are Cauchy sequence):

$$\lim_{j\to\infty}\sup_{k\to\infty} E^1_{jk} = 0.$$

Thanks to (VI), we can rewrite for $v \in V(\partial\mathbb{P})$ the definition of l_j as follow:

$$l_j(v') = -\int_{\partial\mathbb{P}} M(u_j - u)\cdot\nabla_{\partial\mathbb{P}} v' d(\partial\mathbb{P}) - \int_{\mathbb{P}} (u_j - u)\cdot\nabla v' d\mathbb{P} = 0.$$

Combining this with (11.4.23), yields that: $\lim\limits_{j\to\infty} l_j = 0$ in $V(\partial\mathbb{P})^\dagger$. Combining this (11.4.24), yields:

$$\lim_{j\to\infty}\psi_j = 0 \text{ in } V(\partial\mathbb{P}). \tag{11.4.30}$$

Hence $\lim\limits_{j\to\infty}\sup\limits_{k\to\infty} E^2_{j,k} = \lim\limits_{j\to\infty}\sup\limits_{k\to\infty} 2\|(u_j - u_k)\|^2_{L^2(\mathbb{P})^3} + 4C_{M,P}\|(u_j - u_k)_{\partial\mathbb{P}}\|^2_{L^2(\partial\mathbb{P})}$. Combining this with (11.4.23) conclude the proof of (11.4.29). Thanks to (11.4.29) and (11.4.28), we directly get that $(\tilde{u}_j)_j$ is a Cauchy sequence in $H^1(\mathbb{P})^3$. Therefore there exists $\tilde{u} \in H^1(\mathbb{P})^3$ such that:

$$\tilde{u} = \lim_{j\to\infty}\tilde{u}_j \text{ in } H^1(\mathbb{P})^3,$$

and thanks to (11.4.30) and (11.4.25) this becomes:

$$\tilde{u} = \lim_{j\to\infty} u_j \text{ in } L^2(\mathbb{P})^3.$$

Combining this with (11.4.23) yields $u = \tilde{u} \in H^1(\mathbb{P})^3$ which concludes the proof. □

Proof of Lemma 11.4.6. Let M, C and C' satisfying (I). Let M' be an element of $C^1\big(\mathbb{R}^2, \mathcal{M}_2(\mathbb{R}^2)\big)$ such that

$$\|M' - M\|_{C^1(\partial\mathbb{P})} \leq \sqrt{\frac{C'}{2}}.$$

Then thanks to the Leibniz's formula, we can prove that for all $u \in H^1(\mathbb{P})^2$ with $u_{\partial\mathbb{P}} \in H^1(\mathbb{P})^2$ and $\mathrm{div}(u) = 0$, we have:

$$\mathrm{Re}\int_{\partial\mathbb{P}} \mathrm{div}_\Gamma(M' u_{\partial\mathbb{P}})\overline{\mathrm{div}_\Gamma(u_{\partial\mathbb{P}})} d(\partial\mathbb{P}) \geq -\frac{C'}{2}\|u_{\partial\mathbb{P}}\|^2_{H^1(\partial\mathbb{P})} + \mathrm{Re}\int_{\mathbb{R}^2} \mathrm{div}_\Gamma(M u_{\partial\mathbb{P}})\overline{\mathrm{div}_\Gamma(u_{\partial\mathbb{P}})} d(\partial\mathbb{P}).$$

Combining this last line with (I) leads to:

$$-\mathrm{Re}\int_{\partial\mathbb{P}} \mathrm{div}_{\partial\mathbb{P}}(Mu_{\partial\mathbb{P}})\overline{\mathrm{div}_{\partial\mathbb{P}}(u_{\partial\mathbb{P}})}d(\partial\mathbb{P}) \le C_{M,P}\|(Pu)_{\partial\mathbb{P}}\|^2_{H\left(\mathbf{rot}_{\partial\mathbb{P}},\partial\mathbb{P}\right)} + \|Pv\|_{H(\mathbf{rot},\mathbb{P})}$$
$$- \frac{C'}{2}\left(\|u\|^2_{H^1(\mathbb{P})^3} + \|u_{\partial\mathbb{P}}\|^2_{H^1(\partial\mathbb{P})^2}\right). \qquad (11.4.31)$$

Let P' an element of $C^1\big(\mathbb{P},\mathcal{M}_2(\mathbb{R}^3)\big)$ then if $\|P'-P\|_{C^1(\mathbb{P})}$ is small then we have:

$$C\|P'u\|^2_{H(\mathbf{rot},\mathbb{P})} \ge C\|Pu\|^2_{H(\mathbf{rot},\mathbb{P})} - \frac{C'}{4}\|u\|^2_{H^1(\mathbb{P})^3}$$

Adding this last estimate with (11.4.31) which conclude the proof. □

Proof of Proposition 11.4.3. By using truncation function and correction with gradient(see proof of Lemma 10.1.1), we can prove that we can reduce to the case of H^δ satisfying:

$$H^\delta \times n = 0 \text{ on } \partial B_R, \qquad (11.4.32)$$

which conclude the proof of (11.4.17). Let $(\chi_i^N)_{1\le i\le N}$ be an arbitrary smooth partition of unity. Thanks to the Rellich lemma, a sufficient condition of (11.4.16) is the following one: If this last unit partition is fine enough then for all $1 \le i \le N$ the quantity $\chi_i^N H^\delta$ is bounded in $H^1(B_R\cap\Omega)$. Therefore, let $1\le i\le N$ and let us prove that this last property holds for this i.

Assume first that $\mathrm{supp}(\chi_i^N)\cap\Gamma = \emptyset$. In this case we have by the definition of the space X_0^δ that:

$$\mathrm{div}(\chi_i^N H^\delta) = \nabla\chi_i^N\cdot H^\delta + \chi_i^\delta \cancel{\mathrm{div}(H^\delta)} \quad \text{and} \quad \mathbf{rot}(\chi_i^N H^\delta) = \nabla\chi_i^N\cdot H^\delta + \chi_i^\delta\mathbf{rot}(H^\delta),$$

which directly leads that $\chi_i^N H^\delta$ is bounded in the space $H(\mathbf{rot}; B_R\cap\Omega)\cap H(\mathrm{div}; B_R\cap\Omega)$. Moreover this last quantities vanishes in the neighborhood of Γ. Thus thanks to (11.4.32) and [60, Lemma 5.4.2] we directly get that this last quantity is bounded in $H^1(B_R\cap\Omega)$.

Assume now that $\mathrm{supp}(\chi_i^N)\cap\Gamma \ne \emptyset$. Since our manifold Γ is smooth enough then there exists C^∞ diffeomorphism $\Psi_i : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that we have:

$$\Psi_i^{-1}(\mathrm{supp}(\chi_i^N)\cap\Omega) \subset \mathbb{P} \quad \text{and} \quad \Psi_i^{-1}\left(\mathrm{supp}(\chi_i^N)\cap\Gamma\right) \subset \partial\mathbb{P}. \qquad (11.4.33)$$

We refer the reader to Figure 11.1 for an illustration of this map. From this last application we introduce the vector sequence $H_\star^\delta : \mathbb{P}\mapsto\mathbb{C}^3$ defined for $x\in\mathbb{P}$ by:

$$H_\star^\delta(x) := \det\left(\mathrm{D}\,\Psi_i(x)\right)\mathrm{D}\,\Psi_i(x)^{-1}\chi_i(x')H^\delta(x') \quad \text{with} \quad x' := \Psi_i(x)\in\Omega, \qquad (11.4.34)$$

and we now prove by using Lemma 11.4.4 that this last quantity is bounded in $H^1(\mathbb{P})$.

We introduce for convenience the tensor field defined for $x\in\Psi_i^{-1}(\mathrm{supp}(\chi_i^N)$ by:

$$\mu_{\mathrm{eff}}^\star(x) := \mu_{\omega_i}(x)\det\left(\mathrm{D}\,\Psi_i(x)\right)^{-1}\mathrm{D}\,\Psi_i(x)^{-1}\mu_{\mathrm{eff}}(x')\,\mathrm{D}\,\Psi_i(x) \quad \text{with} \quad x' := \Psi_i(x)\in\Omega,$$

where $\mu_{\omega_i} : \partial\mathbb{P}\mapsto\Gamma$ is a smooth function such that for all $F : \mathrm{supp}(\chi_i^N)\cap\Gamma\mapsto\mathbb{C}$ we have:

$$\int_{\Psi_i^{-1}(\mathrm{supp}(\chi_i^N)\cap\Gamma)} F\big(\Psi_i(x)\big)\mu_{\omega_i}(x)dx = \int_{\mathrm{supp}(\chi_i^N)\cap\Gamma} F\big(x_\Gamma\big)dx_\Gamma.$$

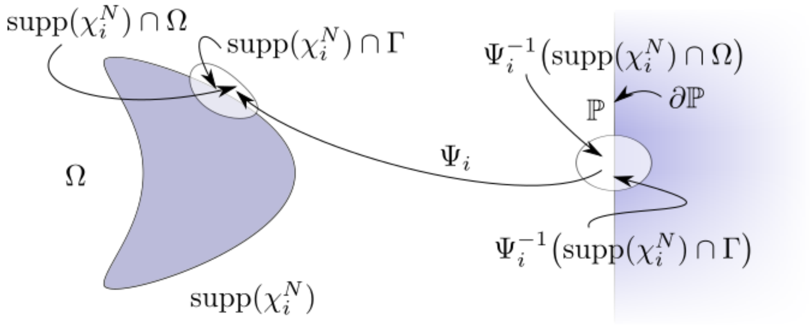


Figure 11.1: Illustration of Ψ_i

We can prove that we can extend the tensor field $\mu^\star_{\text{eff}}$ for all $x \in \partial\mathbb{P}$ and hereafter $\mu^\star_{\text{eff}}$ is this extension.

Thanks to [58, Corrally 3.58] the curl of the following vector field defined for $x \in \mathbb{P}$ by:

$$\det\big(\mathrm{D}\,\Psi_i(x)\big)\,\mathrm{D}\,\Psi_i^\dagger(x)\chi_i(x')H^\delta(x') \quad \text{with} \quad x' := \Psi_i(x),$$

is bounded in $L^2(\mathbb{P})$. Therefore there exists $C > 0$ independent of δ such that:

$$\big\|\mathbf{rot}\big(I_{\Psi_i}H^\delta_\star\big)\big\|_{L^2(\mathbb{P})} \leq C \quad \text{and} \quad \delta^{\frac{1}{2}}\,\big\|\mathrm{rot}_\Gamma\big(I_{\Psi_i}H^\delta_\star\big)\big\|_{L^2(\partial\mathbb{P})} \leq C, \tag{11.4.35}$$

where we defined the tensor field on $\mathbb{P}$ by $I_{\Psi_i} := \mathrm{D}\,\Psi_i^\dagger\,\mathrm{D}\,\Psi_i$.

If we were able to apply Lemma 11.4.4 to the sequence of vector field $(H^\delta_\star)_\delta$ and tensor fields $\mu^\star_{\text{eff}}$ and I_{Ψ_i} then we could conclude our proof. Indeed this result would provide that the sequence fields $(H^\delta_\star)_\delta$ is bounded in $H^1(\mathbb{P})$. Nevertheless, if $H^\delta_\star$ assumption satisfied (VI) then the anti-linear form $l^\delta_i \in V(\mathbb{P})^\dagger$ (the space $V(\mathbb{P})$ defined one the proof of Lemma 11.4.4)

$$\langle l^\delta_i, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})} := \int_{\mathbb{P}} H^\delta_\star \cdot \nabla v + \delta \int_{\partial\mathbb{P}} \mu^\star_{\text{eff}}\big(H^\delta_\star\big)_{\partial\mathbb{P}} \cdot \nabla_{\mathbb{P}} v, \tag{11.4.36}$$

should vanish. To counter this problem, one adds a potential to $H^\delta_\star$:

$$\tilde{H}^\delta_\star := \tilde{H}^\delta_\star + I^{-1}_{\Psi_i}\nabla\phi^\delta_\star \tag{11.4.37}$$

where $\nabla\phi^\delta_\star \in V(\mathbb{P})$ is uniquely defined by the solution of the problem: Find $\phi^\delta_\star \in V(\mathbb{P})$ such that for all $v \in V(\mathbb{P})$

$$\forall v \in V(\mathbb{P}),\ A^\delta(\phi^\delta_\star, v) = -\langle l^\delta_i, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})}, \tag{11.4.38}$$

and $A^\delta : V(\mathbb{P}) \times V(\mathbb{P}) \mapsto \mathbb{C}$ is defined for by for $(u, v) \in V(\mathbb{P})^2$ by:

$$A^\delta(u, v) := \int_{\mathbb{P}} I^{-1}_{\Psi_i}\nabla u \cdot \nabla v + \delta \int_{\partial\mathbb{P}} \mu^\star_{\text{eff}} I^{-1}_{\Psi_i}\nabla_{\partial\mathbb{P}} u \cdot \nabla_{\partial\mathbb{P}} v.$$

In the later we prove this last problem is well posed when the diameter of $\text{supp}(\chi_i^N)$ is small enough. Then one directly get that $\tilde{H}^\delta_\star$ satisfies:

$$\forall v \in V(\mathbb{P}),\ \int_{\mathbb{P}} \tilde{H}^\delta_\star \cdot \nabla v + \delta \int_{\partial\mathbb{P}} \mu^\star_{\text{eff}}\big(\tilde{H}^\delta_\star\big)_{\partial\mathbb{P}} \cdot \nabla_{\mathbb{P}} v = 0. \tag{11.4.39}$$

To prove that (11.4.38) is well posed, we now apply the Lax Milgram theorem. Thus we now prove the two following proposition:

- If the diameter of the support of the function χ_i^N is small enough then the sesquilinear form A^δ is uniformly to the small parameter δ coercive on the space $V(\mathbb{P})$ in the sense that there exists a constant $C > 0$ independent of δ such that for all $u \in V(\mathbb{P})$ we have:
$$\mathrm{Re} A^\delta(u,u) \geq C \left(\|\nabla u\|^2_{L^2(\mathbb{P})} + \delta \|\nabla_{\partial\mathbb{P}} u\|^2_{L^2(\mathbb{P})} \right). \tag{11.4.40}$$
- The anti-linear form l_i^δ is well an element of $V(\mathbb{P})^\dagger$.

Thanks to the regularity of our boundary Γ and the regularity of the tensor field μ_{eff} on Γ we have for all $\eta > 0$ that: If the diameter of the support of the function χ_i^N is small enough then we can chose the application Ψ_i such that:

$$\|I_{\Psi_i} - I\|_{C^1(\mathbb{P})} \leq \eta \quad \text{and} \quad \|\mu^\star_{\text{eff}} - \mu^\star_0\|_{C^1(\partial\mathbb{P})} \leq \eta, \tag{11.4.41}$$

where $\mu_0^\star$ is a constant positive definite hermitian matrix. Thus if η is small enough then the matrix $I_{\Psi_i}^{-1}$ and $\mu^\star_{\text{eff}} I_{\Psi_i}^{-1}$ are definite positive and then we can easily conclude the proof of (11.4.40). Now let us prove that l_i^δ is well continuous. Indeed, thanks to (11.4.15) , the smoothness of the map Ψ_i and (11.4.34) we have existence of $C > 0$ independent of δ such that:
$$\sqrt{\|H^\delta_\star\|^2_{L^2(\mathbb{P})^3} + \delta \|\left(H^\delta_\star\right)_{\partial\mathbb{P}}\|^2_{L^2(\partial\mathbb{P})^2}} < C.$$
Combining this with (11.4.36), yields that for all $v \in V(\mathbb{P})$ we have:

$$|\langle l_i^\delta, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})}| \leq \sqrt{\|\nabla v\|^2_{L^2(\mathbb{P})^3} + \delta \|\nabla_{\mathbb{P}} v\|^2_{L^2(\partial\mathbb{P})^2}} C. \tag{11.4.42}$$

Therefore we concluded the proof of the continuity of l_i^δ. Hence we well can apply the Lax Milgram theorem and then $\phi^\delta_\star$ is well defined and (11.4.39) holds.

Now let us prove that $\nabla \phi^\delta_\star \in H^1(\mathbb{P})$ and $\nabla_{\partial\mathbb{P}} \phi^\delta_\star \in H^1(\partial\mathbb{P})$ with the existence of $C > 0$ such that we have:
$$\|\nabla \phi^\delta_\star\|_{H^1(\mathbb{P})^2} + \delta^{\frac{1}{2}} \|\nabla_{\partial\mathbb{P}} \phi^\delta_\star\|_{H^1(\partial\mathbb{P})^2} \leq C. \tag{11.4.43}$$
Combining (11.4.40) with (11.4.42), yields the existence of $C > 0$ independent of δ such that:
$$\|\nabla \phi^\delta_\star\|_{L^2(\mathbb{P})} + \delta^{\frac{1}{2}} \|\nabla_{\partial\mathbb{P}} \phi^\delta_\star\|_{L^2(\partial\mathbb{P})} \leq C. \tag{11.4.44}$$
Now let us prove the existence of functions $L_0^\delta : \mathbb{P} \mapsto \mathbb{C}$ and $L_1^\delta : \partial\mathbb{P} \mapsto \mathbb{C}$ such that for all $v \in V(\mathbb{P})$ we have:

$$\langle l_i^\delta, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})} = \int_{\mathbb{P}} L_0^\delta \cdot \overline{v} + \delta \int_{\partial\mathbb{P}} L_1^\delta \cdot \overline{v}, \tag{11.4.45}$$

with the existence of $C > 0$ independent of δ such that we have

$$\|L_0^\delta\|_{L^2(\mathbb{P})} \leq C \quad \text{and} \quad \delta^{\frac{1}{2}} \|L_1^\delta\|_{L^2(\mathbb{P})} \leq C. \tag{11.4.46}$$

Moreover the support of this last function satisfies:

$$\mathrm{supp}(L_0^\delta) \subset \Psi_i^{-1}(\mathrm{supp}(\chi_i^N) \cap \Omega) \quad \text{and} \quad \mathrm{supp}(L_1^\delta) \subset \Psi_i^{-1}(\mathrm{supp}(\chi_i^N) \cap \Gamma). \tag{11.4.47}$$

Indeed, for all $v \in V(\mathbb{P})$, we have:

$$
\begin{aligned}
\langle l_i^\delta, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})} &= \int_{\mathbb{P}} H_\star^\delta \cdot \nabla v + \delta \int_{\partial\mathbb{P}} \mu_{\text{eff}}^\star \big(H_\star^\delta\big)_{\partial\mathbb{P}} \cdot \nabla_{\mathbb{P}} v \\
&= \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Omega)} H_\star^\delta \cdot \nabla(\tilde{v}\circ\Psi_i) + \delta \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Gamma)} \mu_{\text{eff}}^\star \big(H_\star^\delta\big)_{\partial\mathbb{P}} \cdot \nabla_{\mathbb{P}}(\tilde{v}\circ\Psi_i), \\
&= \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Omega)} H_\star^\delta \cdot \mathrm{D}\,\Psi_i^\dagger \cdot \nabla\tilde{v}\circ\Psi_i + \delta \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Gamma)} \mu_{\text{eff}}^\star \big(H_\star^\delta\big)_{\partial\mathbb{P}} \cdot \mathrm{D}\,\Psi_i^\dagger \cdot \nabla_{\mathbb{P}}\tilde{v}\circ\Psi_i, \\
&= \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Omega)} \mathrm{D}\,\Psi_i \cdot H_\star^\delta \cdot \nabla\tilde{v}\circ\Psi_i + \delta \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Gamma)} \mathrm{D}\,\Psi_i \cdot \mu_{\text{eff}}^\star \big(H_\star^\delta\big)_{\partial\mathbb{P}} \cdot \nabla_{\mathbb{P}}\tilde{v}\circ\Psi_i.
\end{aligned}
$$

Replacing $H_\star^\delta$ and $\mu_{\text{eff}}^\star$ by their definition in these last line leads to:

$$
\begin{aligned}
\langle l_i^\delta, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})} &= \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Omega)} \det\big(\mathrm{D}\,\Psi_i\big)\big(\chi_i H^\delta \cdot \nabla\tilde{v}\big)\circ\Psi_i + \delta \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Gamma)} \mu_{\omega_i}\Big(\mu_{\text{eff}}\chi_i H_\Gamma^\delta \cdot \nabla_\Gamma\tilde{v}\Big)\circ\Psi_i, \\
&= \int_\Omega \chi_i H^\delta \cdot \nabla\tilde{v} + \delta \int_\Gamma \mu_{\text{eff}}\chi_i H_\Gamma^\delta \cdot \nabla_\Gamma\tilde{v}, \\
&= \cancel{\int_\Omega H^\delta \cdot \nabla(\chi_i\tilde{v}) + \delta\int_\Gamma \mu_{\text{eff}} H_\Gamma^\delta \cdot \nabla_\Gamma(\chi_i\tilde{v})} + \int_\Omega (\nabla\chi_i \cdot H^\delta)\overline{\tilde{v}} + \delta\int_\Gamma (\mu_{\text{eff}}\chi_i H_\Gamma^\delta \cdot \nabla_\Gamma\chi_i)\overline{\tilde{v}}, \\
&= \int_{(\operatorname{supp}(\chi_i^N)\cap\Omega} (\nabla\chi_i \cdot H^\delta)\overline{\tilde{v}} + \delta\int_{\operatorname{supp}(\chi_i^N)\cap\Gamma} (\mu_{\text{eff}}\chi_i H_\Gamma^\delta \cdot \nabla_\Gamma\chi_i)\overline{\tilde{v}}, \\
&= \int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Omega)} \det\big(\mathrm{D}\,\Psi_i\big)(\nabla\chi_i \cdot H^\delta)\circ\Psi_i\overline{v} + \delta\int_{\Psi_i^{-1}(\operatorname{supp}(\chi_i^N)\cap\Gamma)} \mu_{\omega_i}(\mu_{\text{eff}}\chi_i H_\Gamma^\delta \cdot \nabla_\Gamma\chi_i)\circ\Psi_i\overline{v},
\end{aligned}
$$

which conclude the proof of (11.4.45) and (11.4.47) if we chose:

$$
L_0^\delta := \det\big(\mathrm{D}\,\Psi_i\big)(\nabla\chi_i \cdot H^\delta)\circ\Psi_i \quad \text{and} \quad L_1^\delta := \mu_{\omega_i}(\mu_{\text{eff}}\chi_i H_\Gamma^\delta \cdot \nabla_\Gamma\chi_i)\circ\Psi_i.
$$

The estimate (11.4.46) are direct consequences of (11.4.15) and the definition of L_0^δ and L_1^δ.

Thanks to the smoothness of the maps μ_{eff} and I_{Ψ_i}, by using similar argument as the proof of [57, Theorem 4.21], we can prove that for all $j \in \{1,2\}$ we have:

$$
\partial_{x_j}\phi_\star^\delta \in V(\mathbb{P}),
$$

where (x_1, x_2, x_3) is the variable of the map $\phi_\star^\delta$. Thus for all $\phi \in \overline{\mathbb{P}}$, taking $\partial_{x_j}\phi$ as a test function in (11.4.38) yields:

$$
A^\delta(\partial_{x_j}\phi_\star^\delta, \phi) = \langle l_{i,j}^\delta, \phi\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})}, \tag{11.4.48}
$$

where we defined the anti-linear form $l_{i,j}^\delta$ for $v \in V(\mathbb{P})$ by:

$$
\langle l_{i,j}^\delta, \phi\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})} := \int_{\mathbb{P}} \partial_{x_j}(I_{\Psi_i}^{-1})\nabla u\cdot\nabla v + \delta\int_{\partial\mathbb{P}} \partial_{x_j}(\mu_{\text{eff}}^\star I_{\Psi_i}^{-1})\nabla_{\partial\mathbb{P}} u\cdot\nabla_{\partial\mathbb{P}} v - \int_{\mathbb{P}} L_0^\delta\cdot\partial_{x_j}\overline{v} - \delta\int_{\partial\mathbb{P}} L_1^\delta\cdot\partial_{x_j}\overline{v}.
$$

Thanks to the smoothness of the maps μ_{eff} and I_{Ψ_i} (11.4.46) and (11.4.44), we get the existence of $C > 0$ independent of δ such that for all $v \in V(\mathbb{P})$ we have:

$$
|\langle l_{i,j}^\delta, v\rangle_{V(\mathbb{P})^\dagger - V(\mathbb{P})}| \leq \sqrt{\|\nabla v\|_{L^2(\mathbb{P})^3}^2 + \delta\|\nabla_{\mathbb{P}} v\|_{L^2(\partial\mathbb{P})^2}^2}\,C.
$$

Combining this with (11.4.48) and (11.4.40), yields the existence of $C > 0$ independent of δ such that for all $j \in \{1, 2\}$ we have:

$$\|\nabla \partial_{x_j} \phi_\star^\delta\|^2_{L^2(\mathbb{P})^3} + \delta \|\nabla_{\mathbb{P}} \partial_{x_j} \phi_\star^\delta\|^2_{L^2(\partial\mathbb{P})^2} < C. \tag{11.4.49}$$

Moreover, by using there is an argument in the proof of [57, Theorem 4.21] which state that combining (11.4.49) and

$$-\mathrm{div}\big(I_{\Psi_i}^{-1} \nabla \phi_\star^\delta\big) = L_0^\delta \text{ in } \mathbb{P},$$

yields $\partial^2_{x_3} \phi_\star^\delta \in L^2(\mathbb{P})$ the existence of $C > 0$ such that:

$$\|\partial^2_{x_3} \phi_\star^\delta\|_{L^2(\mathbb{P})} \leq C.$$

Combining this with (11.4.49) concludes the proof of (11.4.43).

(11.4.45) and (11.4.46) are sufficient condition to have Thus we succeed to properly construct the function $\nabla \psi_\star^\delta$ and the required property (11.4.39) is a direct consequence of (11.4.38).

Thanks to (11.4.35), (11.4.43), the identity identity $\mathbf{rot}\nabla = 0$ and $\mathbf{rot}_{\partial\mathbb{P}} \nabla_{\partial\mathbb{P}} = 0$, we directly get the existence of $C > 0$ independent of δ such that the following estimate holds:

$$\begin{cases} \left\|\mathbf{rot}\big(I_{\Psi_i} \tilde{H}_\star^\delta\big)\right\|_{L^2(\mathbb{P})} + \left\|\tilde{H}_\star^\delta\right\|_{L^2(\mathbb{P})} \leq C, \\ \delta^{\frac{1}{2}} \left(\left\|\mathrm{rot}_\Gamma\big(I_{\Psi_i} \tilde{H}_\star^\delta\big)\right\|_{L^2(\partial\mathbb{P})} + \left\|\tilde{H}_\star^\delta\right\|_{L^2(\partial\mathbb{P})}\right) \leq C. \end{cases} \tag{11.4.50}$$

Thus thanks to (11.4.50) and (11.4.39), $\tilde{H}_\star^\delta$ satisfies (V) and (VI). Moreover we have seen that we can have η as small as we want in (11.4.41). Therefore to to apply Lemma 11.4.4 to $\tilde{H}_\star^\delta$ it remains to prove that the tensor I_{Ψ_i} and $\mu^\star_{\mathrm{eff}}$ satisfies (I), (III) and (IV)

Indeed, thanks to Lemma 11.4.5, since $\mu_0^\star$ is a positive constant hermitian matrix, $\mu_0^\star$ and $\mathbb{I}_3$ satisfies (I), (III) and (IV). Moreover, since we have seen that we can get η as small as we want in (11.4.41) we can apply Lemma 11.4.6 which yields that I_{Ψ_i} and $\mu^\star_{\mathrm{eff}}$ also satisfies (I), (III). The property (IV) is a consequence of (11.4.33). We can easily prove that I_{Ψ_i} and $\delta\mu^\star_{\mathrm{eff}}$ also satisfies (I), (III) and (IV), by replacing $C_{M,P}$ by $C_{\delta\mu^\star_{\mathrm{eff}}, I_{\Psi_i}} := \delta C_{M,P}$. Therefore thanks to (11.4.50) and Lemma 11.4.4, $\tilde{H}_\star^\delta \in H^1(\mathbb{P})^3$ and there exists $C > 0$ independent of $\delta > 0$ such

$$\|\tilde{H}_\star^\delta\|_{H^1(\mathbb{P})^3} \leq C.$$

Combining this with (11.4.43) and (11.4.37) conclude the proof of $\tilde{H} \in H^1(\mathbb{P})$ and $\|\tilde{H}\|_{H^1(\mathbb{P})} \leq C$. Therefore we can now conclude our proof. □

Corollary 11.4.7. *We have the following convergence:*

$$\lim_{\delta\to 0} \|H^\delta\|_{L^2(B_R \cap \Omega)} = \lim_{\delta\to 0} \delta^{\frac{1}{2}} \|H_\Gamma^\delta\|_{L^2(\Gamma)} = \|G_e^2(n \times H^\delta)\|_{H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R}; \partial B_R)} = 0.$$

Proof. Thanks to Proposition 11.4.3 a sufficient condition is to prove that H^δ weakly converge to 0 in the space $H(\mathbf{rot}; B_R \cap \Omega)$ and $\delta^{\frac{1}{2}} H^\delta$ weakly converge to 0 in $L^2(\Gamma)$.

Indeed let H' be a smooth vector field then thanks to the regularity of the coefficient $\epsilon_{\text{eff}}^{-1}$ and μ_{eff} we have:

$$\vec{\mathbf{rot}}_\Gamma\left(\epsilon_{\text{eff}}^{-1}\text{rot}_\Gamma H'\right) - \mu_{\text{eff}} H' \in H^{-\frac{1}{2}}(\text{div}_{\partial B_R}; \partial B_R).$$

Therefore the following quantity:

$$\int_\Gamma \epsilon_{\text{eff}}^{-1}\text{rot}_\Gamma H^\delta \cdot \text{rot}_\Gamma H' - \int_\Gamma \left(\mu_{\text{eff}} H^\delta, H'\right)$$

is bounded in δ which leads to:

$$\lim_{\delta\to 0} \delta \int_\Gamma \epsilon_{\text{eff}}^{-1}\text{rot}_\Gamma H^\delta \cdot \text{rot}_\Gamma H' - \delta \int_\Gamma \left(\mu_{\text{eff}} H^\delta, H'\right) = 0.$$

Thus combining this last convergence with (11.4.12) leads to:

$$\lim_{\delta\to 0} a_0(H^\delta, H') = 0. \tag{11.4.51}$$

Moreover the sequence H^δ is bounded in $H(\mathbf{rot}; B_R \cap \Omega)$ and then this last sequence converge up to a sub-sequence. We emphasize that (11.4.51) is true for all smooth vector H' and then we directly yields that the the weak limit of $(H^\delta)_{\delta>0}$ is zero. Moreover the sequence $(H^\delta)_{\delta>0}$ is bounded in $H^{-\frac{1}{2}}(\text{div}_{\partial B_R}; \partial B_R)$ which leads to $(\delta^{\frac{1}{2}} H^\delta)_{\delta>0}$ converge to zero in the space $H^{-\frac{1}{2}}(\text{div}_{\partial B_R}; \partial B_R)$. Thus using that this last sequence is bounded in $L^2(\Gamma)$ conclude the proof of the weak convergence $(\delta^{\frac{1}{2}} H^\delta)_{\delta\to 0}$ to zero in the space $L^2(\Gamma)$.□

Proof of Lemma 11.4.1. We have the following decomposition:

$$a_1^\delta(H^\delta, H^\delta) = C^\delta + K^\delta,$$

where we defined:

$$\begin{cases} C^\delta := \displaystyle\int_{B_R\cap\Omega} |\mathbf{rot} H^\delta|^2 \text{d}\Omega + \delta \int_\Gamma \epsilon_{\text{eff}}^{-1} |\text{rot}_\Gamma H^\delta|^2 \text{d}\Gamma + \langle ikG_e^1(n \times H^\delta), (n \times H^\delta) \times n\rangle, \\ K^\delta := -k^2 \displaystyle\int_{B_R\cap\Omega} |H^\delta|^2 - \delta \int_\Gamma (\mu_{\text{eff}} H^\delta, H^\delta)\text{d}\Gamma + \langle ikG_e^2(n \times H^\delta), (n \times H^\delta) \times n\rangle \end{cases}$$

Thus taking $H' = H^\delta$ in (11.4.12) yields: $\lim\limits_{\delta\to 0} C^\delta + K^\delta = 0$. On the other thanks to Corollary 11.4.7 we have $\lim\limits_{\delta\to 0} K^\delta = 0$ which leads to:

$$\lim_{\delta\to 0} C^\delta = 0. \tag{11.4.52}$$

Thanks to [58, Lemma 10.5,Theorem 10.6] we have:

$$\text{Re}\langle ikG_e^2(n \times H^\delta), (n \times H^\delta) \times n\rangle \geq 0.$$

Moreover we recall that the function $\epsilon_{\text{eff}}^{-1}$ is inferiorly bounded by a strictly positive constant. Therefore (11.4.52) leads to:

$$\lim_{\delta\to 0} \|\mathbf{rot}(H^\delta)\|_{L^2(B_R\cap\Omega)} = \lim_{\delta\to 0} \|\text{rot}_\Gamma(H^\delta)\|_{L^2(\Gamma)} = 0.$$

Combining these last convergence with Corollary 11.4.7 leads to:

$$\lim_{\delta\to 0} \|H^\delta\|_{V^\delta} = 0,$$

which contradict (11.4.12) and then conclude the proof. □

Conclusion et perspective

A l'issue de cette thèse, il est intéressant de faire le point sur le travail réalisé. La définition de la $\psi_\Gamma - \delta -$périodicité, nous a apporté une généralisation de la périodicité pour des fonctions définie une surface Γ. Cette définition n'est pas intrinsèque à la surface Γ car elle dépend du choix de l'application ψ_Γ. Cependant cette définition nous apporte une modélisation de couche mince fortement hétérogène: si l'on se donne une fonction très oscillante (donnée du problème) $\tilde{\mu}$ dans la couche mince et que l'on arrive à identifier un δ, fonction ψ_Γ et une fonction μ^δ qui est $\psi_\Gamma - \delta -$périodique tel que $\mu^\delta \approx \tilde{\mu}$. Nous pourrons alors utiliser notre modèle de couche $\psi_\Gamma - \delta -$périodique. Les simulation numériques de couches minces $\psi_\Gamma - \delta$ périodiques sont coûteuses car il faut mailler à l'échelle du petit paramètre δ. Nous avons réussi durant cette thèse à construire des approximations d'ordre 1 pour les équations de Maxwell et d'ordre 2 pour l'équation de Helmholtz . La couche mince très hétérogène est alors remplacée par une condition d'impédance. Les coefficients apparaissant dans les opérateurs d'impédance dépendent du choix de l'application ψ_Γ . Ces approximations nous permettent de réduire le temps de calcul car l'implémentation de ces conditions ne requiert pas de maillage aussi fin que ceux des couches minces $\psi_\Gamma - \delta -$périodiques. Nous avons réussi à valider numériquement nos approximations dans le cas 2D. Le temps de calcul pour la résolution du problème approché est très inférieur à celui nécessaire pour la résolution du problème initial : ainsi le calcul numérique de la solution exacte a pour ordre de grandeur d'une journée et celui du models approchée a pour ordre de grandeur de une heure.

Donnons pour terminer quelques perspectives de notre travail. Tout d'abord, une première perspective est l'implémentation, puis la validaion, d'une méthode numérique pour l'approximation des conditions d'impédance en 3D (Helmholtz et Maxwell). Nous avons déjà mis au point un code scalaire en 3d qui calcule la solution exacte ainsi que la solution approchée avec la conditions aux limites équivalente d ordre 1 lorsque la surface Γ est homéomorphe à un tore. Les résultats correspondant n'on pu être présentés dans cette thèse car il y a encore des bugs dans ces codes : les deux programmes sont déjà implémentés mais nous sommes actuellement en cours de débogage. Une deuxième perspective de notre travail est d'étudier le cas de couche mince $\psi_\Gamma - \delta -$périodique contenant des métamatériaux (changement de signe des coefficients) ou des matériaux fortement conducteurs (coefficient de l'ordre de $1/\delta^2$). On s'attend à observer dans ces deux cas des phénomènes de résonance dans la couche mince. La dernière perspective est d'étendre notre travail à des surfaces Γ contenant des coins afin d'affaiblir nos hypothèses de régularité sur la surface de l'obstacle.

Nomenclature

$\mathbb{H}_0(\hat{Y}_\infty)$ $\left\{u \in \mathbb{H}(\hat{Y}_\infty),\ u = 0 \text{ on } \partial\hat{\Omega}\right\}$, page 211

S^δ $H^1(B_R \cap \Omega) \cap H^1(\Gamma)$, page 249

X_0^δ $\{H \in V^\delta, a_1^\delta(H, \nabla\psi) = 0,\ \forall \psi \in S^\delta\}$, page 249

$H^{-\frac{1}{2}}(\mathrm{rot}_{\partial B_R}; \partial B_R)$ $\left\{u \in H^{-\frac{1}{2}}(\partial B_R)^3, u \text{ is tangential and } \mathrm{rot}_{\partial B_R}(u) \in H^{-\frac{1}{2}}(\partial B_R)\right\}$, page 234

X $\{u \in H(\mathbf{rot}, \Omega \cap B_R), u \times n = 0 \text{ on } \Gamma\}$, page 233

$(\hat{x}, \hat{\nu})$ The microscopic variables, page 18

$(e^i(x_\Gamma))_i$ The dual basis of $(e_(x_\Gamma))_i$, page 124

$(e_i(x_\Gamma))_i$ A basis of $T_{x_\Gamma}\Gamma$, page 124

(x_Γ, ν) The local coordinates, page 20

$\hat{Y}_\infty$ $]0,1[^2\times]-1,\infty[$, page 54

$\hat{Y}_-$ $]0,1[^2\times]-1,0[$, page 64

$\hat{Y}_+$ $]0,1[^2\times]0,\infty[$, page 64

$\boldsymbol{\epsilon}_{\text{eff}}^{-1}(x_\Gamma)$ Scalar to used to define $\mathcal{Z}_1$, page 246

$\boldsymbol{\mu}_{\text{eff}}(x_\Gamma)$ Element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ used to define $\mathcal{Z}_1$, page 246

$\boldsymbol{\rho}_{eff}^1(x_\Gamma)$ Element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ used to define $\mathcal{Z}_2$, page 142

$\boldsymbol{M}_0^\rho(x_\Gamma)$ Element of $T_{x_\Gamma}\Gamma$ used to compute $\boldsymbol{\rho}_{eff}^0$, page 125

$\boldsymbol{M}_{1,0}^\rho$ Element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ used to compute $\boldsymbol{M}_1^\rho$, page 137

$\boldsymbol{M}_{1,3}^\rho$ Element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ used to compute $\boldsymbol{M}_1^\rho$, page 139

$\boldsymbol{M}_1^\rho(x_\Gamma)$ Element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ used to define $\boldsymbol{\rho}_{eff}^1$, page 142

$\boldsymbol{N}_1^\rho(x_\Gamma)$ Element of $\mathcal{L}(T_{x_\Gamma}\Gamma)$ used to define $\boldsymbol{\rho}_{eff}^1$, page 142

δ The small parameter, page 16

δ_Σ Dirac distribution on the surface Σ, page 61

δ_{ij} The Kronecker symbol, page 124

$\mathbb{P}$ $\mathbb{R}^2\times]-\infty,0[$, page 250

Γ_{M} Set where ψ_Γ is smooth, page 28

$\mathrm{D}\,f$ Differential for functions f defined on Γ , page 27

$\mathrm{D}\,X(u_1,u_2)$ The differential for functions defined on a open subset of $\mathbb{R}^2$, page 25

dist The distance function to Γ, page 16

$\widehat{\mathrm{div}}$ Divergence with respect to the microscopic variables, page 47

div_Γ The surface divergence, page 41

$\mathrm{div}_{\mathcal{L}}$ Divergence for functions defined on $\Gamma\times]-\delta,0[$, page 41

DtN Dirichlet to Neumann map on η_0, page 35

$\mathrm{DtN}_{\mathcal{L}}$ Dirichlet to Neumann for functions defined on $\Gamma\times\{\eta_0\}$, page 39

$\mathbb{H}\big(\hat{Y}_\infty\big)$ Functional space of periodic function defined on $\hat{\Omega}$, page 54

$\mathbb{H}_{\mathrm{comp}}\big(\hat{Y}_\infty\big)$ Space of functions of $\mathbb{H}\big(\hat{Y}_\infty\big)$ that vanishe for $\hat{\nu}$ large enough, page 55

$\mathcal{F}(\Gamma\times\hat{Y}_\infty)$ Space of exponentially decreasing vector with respect to $\hat{\nu}$, page 215

$\mathcal{F}_E(\Gamma\times\hat{Y}_\infty)$ Subspace of $\mathcal{F}(\Gamma\times\hat{Y}_\infty)$ for the electro-static problem in $\hat{Y}_\infty$, page 215

$\mathcal{F}_H(\Gamma\times\hat{Y}_\infty)$ Subspace of $\mathcal{F}(\Gamma\times\hat{Y}_\infty)$ for the magneto-static problem in $\hat{Y}_\infty$, page 215

$H^m_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;V(\hat{Y}_\infty)\right)$ Set of function patching admissible, page 54

$C^m_{0,\Gamma_{\mathrm{M}}}\left(\Gamma;V(\hat{Y}_\infty)\right)$ Set of function patching admissible, page 54

$H^{-\frac{1}{2}}(\mathrm{div}_{\partial B_R};\partial B_R)$ $\left\{u\in H^{-\frac{1}{2}}\left(\partial B_R\right)^3, u \text{ is tangential and } \mathrm{div}_{\partial B_R}(u)\in H^{-\frac{1}{2}}\left(\partial B_R\right)\right\}$, page 233

η Positive parameter used to define the near-field zones, the far field zones and the overlapping zones , page 44

η_0 Positive number where $\mathrm{dist}(\mathrm{supp},\Gamma)>\eta_0$, page 34

$\widehat{\nabla}$ Gradient with respect to the microscopic variables, page 47

∇_Γ The surface gradient, page 38

$\nabla_{\mathcal{L}}$ Gradient for functions defined on $\Gamma\times]-\delta,\eta_0[$, page 38

$\hat{\nu}_+$ 0 if $\hat{\nu}\le 0$ and $\hat{\nu}$ if not , page 63

$\hat{\Omega}$ $\mathbb{R}^2\times]-1,\infty[$, page 45

$\mathcal{N}_E$ Solution of the electrostatic problem , page 211

$\mathcal{N}_H$ Kernel of the magneto-static problem, page 213

$\lambda_l(x_\Gamma)$ Norm of $\mathrm{D}\,\psi_\Gamma(x_\Gamma)l,\ l \in \mathbb{Z}^2$, page 72

$\langle\cdot,\cdot\rangle_{\Gamma\times\{\eta_0\}}$ Dual bracket on $H^{\frac{1}{2}}(\Gamma \times \{\eta_0\})$, page 37

$\langle\cdot,\cdot\rangle_{\Sigma_{\eta_0}}$ Dual braket on $H^{\frac{1}{2}}(\Sigma_{\eta_0})$, page 35

$\mathcal{L}$ The local coordinate map, page 20

$\mathcal{M}_1(x_\Gamma)$ scalar used to define $\mathcal{Z}_2$, page 142

$\mathcal{Z}^i_\delta$ $\sum_{j=0}^{i} \delta^j \mathcal{Z}^j$, page 144

$\mathcal{Z}_1$ Impedance operator for the first order condition, page 129

$\mathcal{Z}_2$ Operator used to define $\mathcal{Z}^\delta_2$, page 142

$\mathcal{M}(x_\Gamma, \nu)$ Element of $\mathcal{L}(\mathbb{R}^3)$, page 209

$\mathcal{M}_i$ Terms of the Taylor expansion of $\mathcal{M}(x_\Gamma, \nu)$ with respect to the variable ν, page 209

$\phi_l(x_\Gamma, \hat{x}, \hat{\nu})$ $e^{i2\pi l\hat{x}}e^{-2\pi\lambda_l(x_\Gamma)\hat{\nu}}$, page 72

μ Distribution defined for elements in $\mathbb{H}(\hat{Y}_\infty)^\dagger \oplus \mathbb{C}[\hat{\nu}]$, page 67

Ω The exterior domain, page 16

Ω^δ $\overline{\Omega} \setminus C^\delta$, page 16

Ω_0 $\Gamma\times]0,\eta_0[$, page 121

$\overline{\rho}_0(x_\Gamma), \overline{\mu}_0(x_\Gamma)$ Average on $\hat{Y}_-$ of $\hat{\rho}$ and $\hat{\mu}$, page 128

$\overline{\rho}_1(x_\Gamma), \overline{\mu}_1(x_\Gamma)$ Average on $\hat{Y}_-$ of $2\cdot\hat{\nu}\hat{\rho}(x_\Gamma;\hat{x},\hat{\nu})$ and $2\cdot\hat{\nu}\hat{\mu}(x_\Gamma;\hat{x},\hat{\nu})$, page 142

ϕ_{x_Γ} The chart at x_Γ, page 25

$T_{x_\Gamma}\Gamma$ The Tangent space, page 25

$\mathbf{rot}$ The curl operator, page 200

rot_Γ The scalar surface curl, page 206

$\mathbf{rot}_\Gamma$ Curl in the variable (x_Γ, ν), page 206

$\vec{\mathbf{rot}}_\Gamma$ The vectorial surface curl, page 206

$\mathcal{I}_{\delta,\eta}$ Scaling operator for Electromagnetism, page 231

$\mathcal{I}_\delta$ The scaling operator, page 46

Σ $]0,1[^2\times\{0\}$, page 55

Σ_{η_0} Boundary where DtN is defined, page 35

$\star$ Operator defined for a tensor field and vector field, page 132

$\delta_\Sigma \otimes \delta_\Sigma$ Dirac distribution on $\Sigma \times \Sigma$, page 61

$\mathcal{C}$ Tensor appearing in the definition of a_δ, page 39

$\mathcal{C}^{(k)}(x_\Gamma), c^{(k)}(x_\Gamma)$ Terms appearing in the taylor expansions with respect to ν of $\mathcal{C}$ and C, page 51

γ_n The normal trace, page 245

γ_t, γ_T The tangential traces , page 201

$\mathcal{T}_0$ Elliptic operator with respect to the variables $(\hat{x}, \hat{\nu})$, page 58

$\mathcal{T}^\rho_{k+2}$ Differential operator with respect to the variable x_Γ and $(\hat{x}, \hat{\nu})$, page 52

$\mathcal{T}_k$ Differential operator with respect to the variable x_Γ and $(\hat{x}, \hat{\nu})$, page 52

a^δ Sesquilinear form for the problem posed in C_{δ,η_0}, page 36

a^i_δ Sesquilinear form for the GIBC, page 146

a^δ_1 Sesquilinear form for the 1 order condition GIBC for Electromagnetism, page 248

a_δ Sesquilinear form for the problem posed in $\Gamma \times \{\eta_0\}$, page 39

B_R Open ball of radius R centered at zero, page 232

C Function appearing in the definition of a_δ, page 38

C^δ The thin coating, page 16

C_{δ,η_0} Reduced domain for scattering problem, page 34

$G(x_\Gamma)$ Gaussian curvature, page 38

G_e The Calderon map, page 233

$H(\mathbf{rot};\tilde{\Omega})$ $\left\{u \in L^2(\tilde{\Omega}^\delta)^3,\ \mathbf{rot}(u) \in L^2(\tilde{\Omega}^\delta)^3\right\}$, page 232

$H(x_\Gamma)$ The mean curvature, page 37

$H^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ $\left\{u \in H^{-\frac{1}{2}}(\Gamma)^3, u \text{ is tangential and } \mathrm{div}_\Gamma\, u \in H^{-\frac{1}{2}}(\Gamma)\right\}$, page 248

$H^{-\frac{1}{2}}(\mathrm{rot}_\Gamma;\Gamma)$ $\left\{u \in H^{-\frac{1}{2}}(\Gamma)^3, u \text{ is tangential and } \mathrm{rot}_\Gamma u \in H^{-\frac{1}{2}}(\Gamma)\right\}$., page 248

n Normal unit vector to Ω, page 16

n_δ Normal unit vector to Ω_δ, page 16

O The obstacle, page 16

$R(x_\Gamma)$ The tensor curvature, page 37

u_δ The new unknown defined on $\Gamma\times]-\delta,\eta_0[$, page 34

u_f Intermediate function to define the righ handside of the problem posed in $\Gamma\times]-\delta,\eta_0[$, page 35

V^δ Functional space for the 1 order condition GIBC for Electromagnetism, page 248

$V_{\mathcal{Z}}$ Functional space for the GIBC, page 146

$V_{x_\Gamma}(0)$ Domain of the function ϕ_{x_Γ} , page 25

w^ϵ Solution of the "cell problem", page 211

$w_i(x_\Gamma;\hat{x},\hat{\nu})$ Solutions of the "cell problem", page 124

$W_{x_\Gamma}(x_{x_\Gamma})$ Image of ϕ_{x_Γ}, page 25

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